

The monodromy of F -isocrystals with logarithmic decay

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Abstract

Let U be a smooth geometrically connected affine curve over \mathbb{F}_p with compactification X . Following Dwork and Katz, a p -adic representation ρ of $\pi_1(U)$ corresponds to an F -isocrystal. By work of Tsuzuki and Crew an F -isocrystal is overconvergent precisely when ρ has finite monodromy at each $x \in X - U$. However, in practice most F -isocrystals arising geometrically are not overconvergent and have logarithmic growth at singularities (e.g. characters of the Igusa tower over a modular curve). We give a Galois-theoretic interpretation of these log growth F -isocrystals in terms of asymptotic properties of higher ramification groups.

1 Introduction

1.1 The Riemann-Hilbert correspondence in positive characteristic

The classical Riemann-Hilbert correspondence for a Riemann surface S provides an equivalence of categories between complex representations of $\pi_1(S)$ and holomorphic vector bundles on S with flat connection. In Katz' seminal paper [12] on p -adic modular forms, he proves a mod p analogue of this correspondence. The correspondence is roughly as follows: Fix X to be a proper smooth curve over a finite field k of characteristic p and let D be a finite set of k -points in X . Let Y be the curve $X - D$ and let $j : Y \rightarrow X$ be the natural open immersion. We fix formal schemes \mathcal{X} (resp. \mathcal{Y}) whose special fibers are X (resp. Y). The correspondence then gives an equivalence of categories between p -adic representations of $\pi_1(Y)$ and étale convergent F -isocrystals: vector bundles with connections commuting with a Frobenius on \mathcal{Y}^{an} that satisfy certain convergence conditions (see [4] for a precise definition). In [24] Tsuzuki proves that a representation of $\pi_1(Y)$ has finite monodromy around D if and only if the corresponding convergent F -isocrystal can be extended to a strict neighborhood of \mathcal{Y}^{an} (i.e. it is overconvergent). The purpose of this article is to generalize Tsuzuki's result by explaining the monodromy of a wider class of F -isocrystals that occur geometrically.

When studying the monodromy around a point $x \in D$, it is significantly simpler to work locally. Let F be the fraction field of the completion of $\mathcal{O}_{X,x}$ and let G_F be the absolute Galois group of F . We let \mathcal{O}_K be the Witt vectors of k with fraction field K . Since X is smooth we know that F is isomorphic to $k((T))$. The inclusion $G_F \hookrightarrow \pi_1(Y)$ lets us restrict our attention to representations $\rho : G_F \rightarrow GL_n(\mathbb{Q}_p)$. To see what happens when we localize an F -isocrystal we require a couple of definitions:

$$\mathcal{E} := \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \in K[[T, T^{-1}]] \mid \begin{array}{l} v_p(a_i) \text{ is bounded below and} \\ v_p(a_i) \rightarrow \infty \text{ as } i \rightarrow -\infty \end{array} \right\}$$

$$\mathcal{E}^\dagger := \left\{ f(T) \in \mathcal{E} \mid f(T) \text{ converges on an annulus } 0 < r < |T|_p < 1 \right\}.$$

Note that \mathcal{E} is the completion of the stalk at the special point of $\text{Spec}(\mathcal{O}_K[[T]])$ and \mathcal{E}^\dagger are those functions which converge on a neighborhood of the special point. A function $f(T)$ is in \mathcal{E}^\dagger if the valuations of the a_n grow at least linearly as $n \rightarrow -\infty$. If we translate the equivalence of

categories to our local situation we get correspondences due to Fontaine and Tsuzuki (see [10], [24], and Section 4.4):

$$\left\{ \begin{array}{l} \text{Free modules } M \text{ of rank } d \text{ over } \mathcal{E} \\ \text{with a connection } \nabla \text{ that commutes} \\ \text{with a unit-root Frobenius} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Continuous representations} \\ \rho : G_F \rightarrow GL_d(\mathbb{Q}_p) \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{Free modules } M \text{ of rank } d \text{ over } \mathcal{E}^\dagger \\ \text{with a connection } \nabla \text{ that commutes} \\ \text{with a unit-root Frobenius} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Continuous representations} \\ \rho : G_F \rightarrow GL_d(\mathbb{Q}_p) \\ \text{where the inertia group } I_F \text{ has finite image} \end{array} \right\}.$$

If (ρ, V) is a p -adic G_F -representation, we obtain a (ϕ, ∇) -module over \mathcal{E} by $(\tilde{\mathcal{E}} \otimes_{\mathbb{Q}_p} V)^{G_F}$, where $\tilde{\mathcal{E}}$ is a p -adically complete unramified closure of \mathcal{E} endowed with a derivative and a Frobenius. This (ϕ, ∇) -module over \mathcal{E} descends to a (ϕ, ∇) -module over \mathcal{E}^\dagger if and only if ρ has finite monodromy. Let $\mathbf{e} = (e_1, \dots, e_d)$ be a \mathbb{Q}_p -basis of V and let $\mathbf{a} = (a_1, \dots, a_d)$ be a \mathcal{E} -basis of $(V \otimes \tilde{\mathcal{E}})^{G_F}$. Then there is an invertible matrix of periods A with entries in $\tilde{\mathcal{E}}$ satisfying

$$\mathbf{a} = A\mathbf{e}.$$

We may think of $A^{-1}\mathbf{a}$ as a bases of solutions to the differential equation given by ∇ . One consequence of Tsuzuki's result is that the entries of A have "linear decay" if and only if ρ has finite monodromy (actually Tsuzuki's theorem says more: the entries of A have linear decay and are algebraic over \mathcal{E}^\dagger). In the finite monodromy case there is also a close relation between the ramification breaks of ρ and the radii of convergence of ∇ (see [14] for an overview). The takeaway is that the decay properties of the differential equation around a point x in D is closely related to the monodromy of ρ at x . It is natural to ask how to realize this relation when ρ has infinite monodromy.

1.2 F -isocrystals with logarithmic decay

Let $f : Z \rightarrow Y$ be a proper smooth morphism. Then $\mathcal{F} = R^i f_* \mathbb{Q}_p$ is a \mathbb{Q}_p -lisse sheaf on Y and therefore corresponds to a p -adic representation $\rho_{\mathcal{F}}$ of $\pi_1(Y)$. We call a representation of $\pi_1(Y)$ arising this way *geometric*. An étale F -isocrystal over Y is called *geometric* if the corresponding representation of $\pi_1(Y)$ is geometric. All geometric étale F -isocrystals are the unit-root sub- F -isocrystal of an overconvergent F -isocrystal (see [26] for a more precise definition). Geometric F -isocrystals are generally not overconvergent. For example, let Y be the ordinary locus of a modular curve and let E^{univ} be the universal ordinary elliptic curve over Y . A famous theorem of Igusa states that the action of $\pi_1(Y)$ on the Tate module of E^{univ} has infinite monodromy at every supersingular point (see [11]) and therefore the corresponding F -isocrystal is not overconvergent. We remark that in this example it is possible to choose an "excellent" Frobenius lifting that makes the Frobenius structure overconvergent. Another important example are the hyper-Kloosterman sheaves arising from the Dwork family of hyper-surfaces. In [22] Sperber shows that the F -isocrystals associated to hyper-Kloosterman sheaves cannot admit an overconvergent Frobenius structure.

This means that if we want to study all geometric étale F -isocrystals we must work in a larger category than the category of overconvergent étale F -isocrystals. The correct category turns out to be étale F -isocrystals that have at worst *logarithmic decay* at each ramified point. These log-decay F -isocrystals were first studied by Dwork and Sperber in [9], where they studied the meromorphic continuation of unit-root L -functions (actually Dwork and Sperber only considered log-decay in the Frobenius structure, but one can show that this condition implies log-decay on the differential structure). Dwork and Sperber prove that the unit-root part of an overconvergent F -isocrystal have log-decay, and therefore that all geometric étale F -isocrystals have log-decay. We remark that the category of geometric F -isocrystals is much smaller than the category of

log-decay F -isocrystals. Wan proves in [26] that the unit-root zeta function associated to a geometric étale F -isocrystal has a meromorphic continuation to the entire p -adic plane. Wan also gives examples in [25] of étale F -isocrystals with log-decay whose unit-root zeta function does not admit a meromorphic continuation.

1.2.1 A local description of log-decay

Let us precisely describe what log-decay means when we work locally around $x \in D$. For $r > 0$ consider the ring $\mathcal{E}^r \subset \mathcal{E}$ defined by

$$\mathcal{E}^r := \left\{ \sum_{n=-\infty}^{\infty} a_n X^n \in \mathcal{E} \mid \begin{array}{l} \text{There exists } c \text{ such that} \\ v_p(a_n) - \frac{\log_p(-n)}{r} > c \text{ for } n < 0 \end{array} \right\}.$$

Let ρ be a p -adic representation of G_F and let M be the corresponding (ϕ, ∇) -module over \mathcal{E} given by Fontaine's theory. Then an F -isocrystal has r -log-decay at x if the M descends to a (ϕ, ∇) -module on \mathcal{E}^r . By Theorem 6.4 that this is equivalent to the entries of the period matrix in equation (1) having a log-decay condition. The purpose of this article is to describe a Galois theoretic property of ρ that determines when M descends to a (ϕ, ∇) -module over \mathcal{E}^r . This gives us to a deeper understanding of the monodromy properties of geometric representations of $\pi_1(Y)$.

The Galois-theoretic meaning of the log-decay condition has to do with the interaction between the p -adic Lie filtration and the ramification filtration determined by ρ . We draw inspiration by a theorem of Sen (see [20]), which nicely explains the interaction of these two filtrations for a local field with mixed characteristic. Let K be a local field and let L be a Galois extension of K whose Galois group $G_{L/K}$ is a p -adic Lie group. The condition that $G_{L/K}$ is a p -adic Lie group guarantees a filtration

$$G_{L/K} = G_{L/K}(0) \supset G_{L/K}(1) \supset G_{L/K}(2) \supset \dots,$$

which satisfies

$$\bigcap_n G_{L/K}(n) = \{0\} \quad \text{and,}$$

$$G_{L/K}(n+1) = \{s \in G_{L/K} \mid s = x^p \text{ for } x \in G_{L/K}(n)\}.$$

We also have a ramification filtration on $G_{L/K}$ described in Section 5.

Theorem 1.1. (*Sen*) *Assume that K is a finite extension of \mathbb{Q}_p with ramification index e . There exists $c > 0$ such that*

$$G^{me+c} \subset G(n) \subset G^{ne-c}.$$

This result completely fails for equal characteristic. In general the ramification filtration behaves too erratically to have a reasonable relationship with the Lie filtration. For example, let G be the Galois group of a totally ramified \mathbb{Z}_p -extension of K . Let $s_0 < s_1 < s_2 < \dots$ be the breaks of the ramification filtration of G . When K has characteristic 0 we deduce from Sen's theorem that s_n is approximately ne . However in characteristic p it is possible for any sequence of integers s_i to occur as long as $s_{i+1} \geq ps_i$ ([5, Proposition 15]). The r -log-decay condition will correspond to the \mathbb{Z}_p extensions such that the breaks s_i grow $O(p^{ir})$. More generally, let $\rho : G_F \rightarrow GL(V)$ be a p -adic Galois representation. Let $L \subset V$ be a lattice stable under the action of G_F . We may regard $G = \text{Im}(G_F \rightarrow GL(L))$ as a Galois group for an extension over F that is also a p -adic Lie group. The lattice L gives a p -adic Lie filtration on G by

$$G(n) = \ker(G \rightarrow GL(L/p^n L)).$$

We then define the category $\mathbf{Rep}_{\mathbb{Q}_p}^r(G_F)$ to be the p -adic representations ρ of G_F such $G^{p^{nr}c} \subset G(n)$. Our main local result is

Theorem 1.2. *There is an equivalence of categories*

$$\{ \text{étale } (\phi, \nabla)\text{-modules over } \mathcal{E}^r. \} \longleftrightarrow \{ \mathbf{Rep}_{\mathbb{Q}_p}^r(G_F) \},$$

sending a p -adic representation V to $D^r(V)$. The functor D^r is compatible with the functors D and D^\dagger of Fontaine and Tsuzuki. That is, if V is in $\mathbf{Rep}_{\mathbb{Q}_p}^r(G_F)$ then $D^r(V) \otimes_{\mathcal{E}^r} \mathcal{E} = D(V)$ and if V is in $\mathbf{Rep}_{\mathbb{Q}_p}^\dagger(G_F)$ then $D^\dagger(V) \otimes_{\mathcal{E}^\dagger} \mathcal{E}^r = D^r(V)$.

1.2.2 A global description of log-decay

In the final section we introduce a global notion of r -log-decay F -isocrystals and relate the r -log-decay property to the geometry of representations of $\pi_1(Y)$. We are inspired by Berthelot's notion of overconvergent sheaves (see [4, Section 2.1]). Berthelot defines a sheaf of rings $\mathcal{O}_{\mathcal{Y}^{an}}^\dagger$ on \mathcal{X}^{an} whose sections on any strict neighborhood of \mathcal{Y}^{an} are the analytic functions on \mathcal{Y}^{an} that converge on an outer annulus of the residue disk of each x in D . An convergent F -isocrystal \mathcal{M} on Y , which is a vector bundle on \mathcal{Y}^{an} with a connection and Frobenius satisfying certain convergent conditions, is overconvergent if it extends to a sheaf of $\mathcal{O}_{\mathcal{Y}^{an}}^\dagger$ -modules. By expanding a convergent F -isocrystal \mathcal{M} in terms of a parameter of $x \in D$, we are able to recover a local F -isocrystals over \mathcal{E} (see [6, Section 4] for a thorough description of this localization process). If \mathcal{M} is overconvergent this local F -isocrystal will descent to \mathcal{E}^\dagger .

In order to have a global notion of r -log-decay, we define a sheaf of rings $\mathcal{O}_{\mathcal{Y}^{an}}^r$ on \mathcal{X}^{an} . Similar to Berthelot's construction, the sections of $\mathcal{O}_{\mathcal{Y}^{an}}^r$ on a strict neighborhood of \mathcal{Y}^{an} are the analytic functions on \mathcal{Y}^{an} that have r -log-decay in the residue disk of each x in D . We then say that an F -isocrystal \mathcal{M} on Y has r -log-decay if \mathcal{M} extends to a sheaf of $\mathcal{O}_{\mathcal{Y}^{an}}^r$ -modules. We prove that \mathcal{M} has r -log-decay if and only if locally at each $x \in D$ the corresponding F -isocrystal over \mathcal{E} descends to \mathcal{E}^r . That is, our global definition of r -log-decay is the same as having r -log-decay locally at each point.

We then use the Riemann-Hurwitz-Hasse formula to give a global geometric interpretation of the r -log-decay property. More precisely, let $\rho : \pi_1(Y) \rightarrow GL_d(V)$ be a \mathbb{Q}_p -representation let $L \subset V$ be a stabilized lattice. We let G be the image of $\pi_1(Y)$ and use L to define a p -adic Lie filtration

$$G = G(0) \subset G(1) \subset G(2) \subset \dots$$

This gives an étale pro- p tower over curves

$$Y = Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \dots$$

Let X_n be the compactification of Y_n . The Riemann-Hurwitz-Hasse formula gives a formula for the genus g_n of X_n in terms of the degree d_n and the different $\delta_{X_n/X}$ of the map $X_n \rightarrow X$. The different is determined by the higher ramification groups, which allows us to deduce upper bounds for g_n from our local results. More precisely the F -isocrystal associated to ρ has r -log-decay if and only if the genus to degree ratio $\frac{g_n}{d_n}$ grows $O(p^{nr})$. This is the content of Theorem 7.2.

1.3 Outline of article

This article is divided into seven sections. In Section 2 we develop some basic properties of the log-decay ring \mathcal{E}^r and its unramified extensions. Section 3 contains an overview of the theory of (ϕ, ∇) -modules over various rings of Laurent series. We then define the log-decay period rings $\tilde{\mathcal{E}}^r$ in Section 4 and develop some of their basic properties. In Section 4.4 we introduce the functors D , D^\dagger and D^r using the period rings of the previous section. The sixth section contains an review of the higher ramification groups and several auxiliary lemmas on ramification theory. The proof of Theorem 1.2 is provided in Section 6. Finally, we return to the global situation in section eight: we give a precise global definition of log-decay F -isocrystals and we deduce global geometric statements about pro- p towers of curves corresponding to log-decay F -isocrystals.

1.4 Future work on Frobenius distributions and genus stability

Let ρ be a p -adic representation of $\pi_1(X)$ with finite monodromy and let M_ρ be the corresponding overconvergent F -isocrystal. Berthelot developed a theory of rigid cohomology, which allows coefficients in M_ρ . Crew demonstrated that much of Deligne's l -adic arguments in [8] translate to the p -adic setting, assuming certain finiteness conditions on the rigid cohomology groups (see [7]). For instance, Crew was able to prove an equidistribution theorem for the eigenvalues of the Frobenius analogous to Deligne's l -adic Chebotarev density theory. The finiteness of rigid cohomology has since been proven by Kedlaya (see [15]) using the p -adic monodromy theorem of Andre, Mebkhout, and Kedlaya (see [1], [18], and [13]). This picture is much less complete when one considers the larger category of convergent F -isocrystals. In particular, the equidistribution of Frobenius eigenvalues is known to be false. Deligne's arguments utilize bounds obtained from the Lefschetz trace formula and the Euler-Poincare formula. The latter depends on higher ramification groups. It therefore seems likely that any attempt to study the Frobenius distribution of an F -isocrystal will rely on a thorough understanding the monodromy of this F -isocrystal. One may hope that the r -log-decay property could measure the failure of Frobenius equidistribution. The author is currently investigating this phenomenon.

Let M be an F -isocrystal of rank one with r -log-decay. This corresponds to a \mathbb{Z}_p -tower of curves X_n . By Theorem 7.2 we know that g_n , the genus of X_n , is bounded by $cp^{(r+1)n}$. Recent work of Kusters and Wan prove that g_n is bounded below by a quadratic in p^n (see [16, Corollary 5.3]). In fact they give a precise formula for g_n in terms of Artin-Schreier-Witt theory. In upcoming work we show that if M is the unit-root part of an overconvergent F -isocrystal, then we may take $r = 1$ (this was previously known by Wan and Sperber, as is mentioned in a remark in [26], though it does not appear to be published). It follows that g_n is bounded above and below by quadratics in p^n . It is natural to ask: when is g_n given precisely by a quadratic in p^n for large n ? Such a tower is called *genus-stable* and such towers have been classified by Kusters and Wan in the context of Artin-Schreier-Witt theory. For example, the Igusa tower is genus-stable. One may hope that any rank one F -isocrystal that is the unit-root subspace of an overconvergent F -isocrystal gives rise to a genus-stable tower. This would imply that any \mathbb{Z}_p -tower arising geometrically is genus-stable. In particular, let $A \rightarrow X$ be an Abelian variety such that the Tate module of $A \times_X Y$ is a rank one \mathbb{Z}_p -module. Then one would hope that the tower of curves obtained by adding torsion points to the function field of X is genus-stable. The author is currently investigating these questions.

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2 Local log decay rings

In this section we will establish several algebraic properties about rings of Laurent series whose tails decay logarithmically. Let k be a finite field and let K be the fraction field of the Witt vectors of k . We denote by v_p the valuation on K normalized so that $v_p(p) = 1$. Consider the sets of Laurent series:

$$\mathcal{E}_{T,K} := \left\{ \sum_{i=-\infty}^{\infty} a_i T^i \in K[[T, T^{-1}]] \mid \begin{array}{l} v_p(a_i) \text{ is bounded below and} \\ v_p(a_i) \rightarrow \infty \text{ as } i \rightarrow -\infty \end{array} \right\}$$

$$\mathcal{E}_{T,K}^r := \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \in \mathcal{E}_{T,K} \mid \text{There exists } c \text{ such that } v_p(a_n) - \frac{\log_p(-n)}{r} \geq c \text{ for } n < 0 \right\}.$$

When the choice of T and k is unambiguous we will drop the subscripts and refer to $\mathcal{E}_{T,K}$ (resp $\mathcal{E}_{T,K}^r$) as \mathcal{E} (resp \mathcal{E}^r). The ring $\mathcal{E}_{T,K}$ is a field that comes with a discrete valuation

$$v\left(\sum a_n T^n\right) = \inf(v_p(a_n)).$$

The valuation ring $\mathcal{O}_{\mathcal{E}_{T,K}}$ consists of elements in $\mathcal{E}_{T,K}$ whose coefficients are integral. The residue field is $F := k((T))$.

Following Kedlaya (see [13, Section 2.3]) we will introduce naive partial valuations to help us keep track of the growth of coefficients. For $a(T) \in \mathcal{E}_{T,K}$ we define

$$v_n^{\text{naive}}(a(T)) = \min_{v_p(a_i) \leq n} \{i\},$$

where $a(T) = \sum a_i T^i$. Informally, we may think of $v_n^{\text{naive}}(a(T))$ as the T -adic valuation of the first n terms when we p -adically expand $a(T)$. In particular, it is possible to write

$$a(T) = \sum_{n >> -\infty}^{\infty} q_n(T) p^n,$$

where $q_n(T) \in \mathcal{O}_K((T))$ has T -adic valuation $v_n^{\text{naive}}(a(T))$ and the term with the smallest exponent has a p -adic unit coefficient. These partial valuations satisfy the following inequalities:

$$\begin{aligned} v_n^{\text{naive}}(a(T) + b(T)) &\geq \min\{v_n^{\text{naive}}(a(T)), v_n^{\text{naive}}(b(T))\} \\ v_n^{\text{naive}}(a(T)b(T)) &\geq \min_{i+k=n} \{v_i^{\text{naive}}(a(T)) + v_k^{\text{naive}}(b(T))\}. \end{aligned}$$

Equality is obtained if the minimum is achieved exactly once. The r -log-decay condition translates into a condition about partial valuations:

Lemma 2.1. *Let $a(T) \in \mathcal{E}_{T,K}$. Then $a(T) \in \mathcal{E}_{T,K}^r$ if and only if there exists $d > 0$ such that $v_n^{\text{naive}}(a(T)) \geq -p^{rn}d$ for each n .*

Proof. Let $n > 0$ and let N be the smallest integer such that $v_p(a_N) \leq n$. Then $N = v_n^{\text{naive}}(a(T))$. Fix c such that

$$v_p(a_k) - \frac{\log_p(-k)}{r} \geq c,$$

for all $k < 0$. We may plug in N for k in this inequality and solve to get

$$v_n^{\text{naive}}(a(T)) \geq -p^{r(n-c)}.$$

□

Lemma 2.2. *$\mathcal{E}_{T,K}^r$ is a field.*

Proof. The only thing to check is that this set is closed under multiplication and inverses. Consider two elements of $a(T), b(T) \in \mathcal{E}_{T,K}^r$. After multiplying by powers of p we may assume that both elements are in $\mathcal{O}_{\mathcal{E}_{T,K}}$. Similarly we may multiply both elements by powers of T to ensure that their residue in F is a unit in $k[[T]]$. In particular $v_0^{\text{naive}}(a(T)) = v_0^{\text{naive}}(b(T)) = 0$. Let c be large enough so that

$$v_n^{\text{naive}}(a(T)), v_n^{\text{naive}}(b(T)) > -cp^{rn}.$$

Then we have

$$v_n^{\text{naive}}(a(T)b(T)) \geq \min_{i+j=n} \{v_i^{\text{naive}}(a(T)) + v_j^{\text{naive}}(b(T))\}.$$

When i or j is equal to n then the term in the minimum is at least $-cp^{rn}$. Otherwise we have $0 < i, j < n$, which gives

$$-cp^{ri} + -cp^{rj} \geq -cp^{rn}.$$

Therefore $a(T)b(T) \in \mathcal{E}_{T,K}^r$.

Since $\mathcal{E}_{T,K}$ is a field, we know that $a(T)$ has an inverse $a^{-1}(T)$ contained in $\mathcal{E}_{T,K}$. We only need to check that $a^{-1}(T)$ satisfies the r -log-decay condition. Once again we assume that $a(T) \in \mathcal{O}_{\mathcal{E}}^\times$ and that the residue of $a(T)$ is a unit in $k[[T]]$ by multiplying by some powers of p and T . In particular $v_0^{\text{naive}}(a(T)) = 0$. Let $c > 0$ be large enough so that $v_n^{\text{naive}}(a(T)) \geq -cp^{rn}$. We will prove $v_n^{\text{naive}}(a^{-1}(T)) \geq -cp^{rn}$ by induction on n . When $n = 0$ we find that $v_0^{\text{naive}}(a^{-1}(T)) = 0$ from the multiplicative inequality and the fact that $v_0^{\text{naive}}(1) = 0$. Now assume $v_j^{\text{naive}}(a^{-1}(T)) \geq -cp^{jr}$ for $j < n$. We have

$$v_n^{\text{naive}}(1) = 0 \geq \min_{i+j=n} \{v_i^{\text{naive}}(a(T)) + v_j^{\text{naive}}(a^{-1}(T))\}.$$

When $i = n$ the term in the minimum function is $-cp^{rn}$ and when $0 < i < n$ we have

$$-cp^{ri} + -cp^{rj} \geq -cp^{rn}.$$

Therefore if $v_n^{\text{naive}}(a^{-1}(T)) < -cp^{rn}$ the minimum would only be obtained exactly once, which yields a contradiction. \square

We make $\mathcal{E}_{T,K}^r$ into a valued field by restricting v to $\mathcal{E}_{T,K}^r$ and let $\mathcal{O}_{\mathcal{E}_{T,K}^r}$ denote the valuation ring. Note that $\mathcal{O}_{\mathcal{E}_{T,K}^r}$ consists of the series in $\mathcal{E}_{T,K}^r$ with integral coefficients. The maximal ideal is $p\mathcal{O}_{\mathcal{E}_{T,K}^r}$ and the residue field is F . By a Proposition of Matsuda (see [17, Proposition 2.2]) we see that $(\mathcal{O}_{\mathcal{E}_{T,K}^r}, p\mathcal{O}_{\mathcal{E}_{T,K}^r})$ is a Henselian pair. This allows us to deduce the following Lemma about unramified extensions of $\mathcal{E}_{T,K}^r$.

Lemma 2.3. *Let L be a finite extension of F and let k' be the residue field of L (so that k' is a finite extension of k). Let K' be the unramified extension of K whose residue field is k' . There is a unique unramified extension E of $\mathcal{E}_{T,K}^r$ whose residue field is L . If $U \in E$ reduces to a unifomrizing element of L then $E = \mathcal{E}_{U,K'}$.*

Lemma 2.3 tells us that $\mathcal{E}_{T,K}$ does not depend on the lifting of a parameter of the residue field. Similarly, K only depends on the constants of the residue field. This means that $\mathcal{E}_{T,K}$ only depends on the residue field. We will therefore often refer to $\mathcal{E}_{T,K}^r$ as \mathcal{E}_F^r , to indicate the dependence on the residue field. This will be particularly helpful in Section 6, where we will consider \mathcal{E}_{F_n} for towers of fields F_n over F .

3 (ϕ, ∇) -modules over \mathcal{E}_T , $\mathcal{E}_{T,K}^r$ and \mathcal{E}_T^\dagger

Let R be either \mathcal{E}_T , $\mathcal{E}_{T,K}^r$ or \mathcal{E}_T^\dagger . The aim of this section is to define (ϕ, ∇) -modules over R . Roughly, these are differential modules over R whose derivation is compatible with a Frobenius semi-linear map.

Definition 3.1. *A ring endomorphism σ of R is a Frobenius if it induces the Frobenius morphism on $W(k) \subset R$ and reduces to the Frobenius morphism modulo p :*

$$\sigma(a) \equiv a^q \pmod{p}.$$

A ϕ -module for σ is an R -module M equip with a σ -semilinear endomorphism $\phi : M \rightarrow M$ whose linearization is an isomorphism. More precisely, we have $\phi(am) = \sigma(a)\phi(m)$ for $a \in R$ and $\sigma^\phi : R \otimes_\sigma M \rightarrow M$ is an isomorphism. We say that M is étale or unit-root if the slopes, in the sense of Dieudonn-Manin, are all zero (see [13, Section 5.2] for a more thorough explanation).*

A Frobenius σ on \mathcal{E} descends to $\mathcal{E}_{T,K}^r$ (resp. \mathcal{E}_T^\dagger) if and only if $\sigma(T) \in \mathcal{E}_{T,K}^r$ (resp. \mathcal{E}_T^\dagger). Some examples are the maps induced by $T \rightarrow T^q$ or $T \rightarrow (1+T)^q - 1$.

Definition 3.2. Let Ω_R be the module of differentials of R over K . In particular $\Omega_R = R dT$. We define the $\delta_T : R \rightarrow \Omega_R$ to be the derivative $\frac{d}{dT}$. A ∇ -module over R is an R module M equip with a connection. That is, M comes with a K -linear map $\nabla : M \rightarrow \Omega_R$ satisfying the Liebnitz rule: $\nabla(am) = \delta_T(a)m + a\nabla(m)$.

We may view a ∇ -module M over R as a differential equation over R by considering the equation $\nabla(x) = 0$. If M is free of rank d we may view it as a first order differential equation in d variables or a d -th order differential equation in one variable by picking a cyclic vector. Now we may introduce (ϕ, ∇) -modules, which is roughly an R -module with ϕ and ∇ structures that are compatible.

Definition 3.3. By abuse of notation, define $\sigma : \Omega_R \rightarrow \Omega_R$ be the map induced by pulling back the differential along σ . In particular

$$\sigma(f(T)dT) = \sigma(f(T))d\sigma(T).$$

A (ϕ, ∇) -module M is an R -module that is both a ϕ -module and a ∇ -module with the following compatibility condition:

$$\begin{array}{ccc} M & \xrightarrow{\nabla} & M \otimes \Omega_R \\ \downarrow \phi & & \downarrow \phi \otimes \sigma \\ M & \xrightarrow{\nabla} & M \otimes \Omega_R. \end{array}$$

We denote the category of (ϕ, ∇) -modules over R by $\mathbf{M}\Phi_{R,\sigma}^\nabla$. An (ϕ, ∇) -module is called *étale* if all of its slopes are 0. We denote the subcategory of $\mathbf{M}\Phi_{R,\sigma}^\nabla$ consisting of étale (ϕ, ∇) -modules as $\mathbf{M}\Phi_{R,\sigma}^{\text{ét},\nabla}$.

4 Period rings with growth properties

4.1 Large period rings and partial valuations

Let $\tilde{\mathbf{A}}$ denote the ring of Witt vectors over the completion of F^{alg} and let $\tilde{\mathbf{B}} = \tilde{\mathbf{A}}[\frac{1}{p}]$. We will let \mathbf{Frob} denote the Frobenius endomorphism of $\tilde{\mathbf{A}}$. Each $x \in \tilde{\mathbf{B}}$ can be written as

$$\sum_{n \gg -\infty}^{\infty} [x_n]p^n,$$

where $[x_n]$ is the Teichmüller lift of $x_n \in F^{\text{alg}}$. We define maps $w_k : \tilde{\mathbf{A}} \rightarrow \mathbb{R} \cup \infty$ by $w_k(x) = \min_{n \leq k} v_T(x_n)$. These maps satisfy the following inequalities:

$$w_k(x+y) \geq \min(w_k(x), w_k(y)) \tag{1}$$

$$w_k(xy) \geq \min_{i+j \leq k} (w_i(x) + w_j(y)). \tag{2}$$

If the minimum is obtained exactly once there is equality. For large k these are related to the naive partial valuations v_k (see [13]). We may also define maps $w_k : M_{d \times e}(\tilde{\mathbf{A}}) \rightarrow \mathbb{R} \cup \infty$ that send the matrix $A = (a_{i,j})$ to $\min(w_k(a_{i,j}))$. Equivalently, if A has the Teichmüller expansion

$$\sum_{n \gg -\infty}^{\infty} [A_n]p^n,$$

with $A_n \in M_{d \times e}(F^{alg})$, then $w_k(A) = \min_{n \leq k} v_T(A_n)$, where $v_T(A_n)$ is the smallest valuation occurring in the entries of A_n . For $A \in M_{d \times e}(\tilde{\mathbf{A}})$ and $B \in M_{e \times f}(\tilde{\mathbf{A}})$ we have the same inequalities:

$$\begin{aligned} w_k(A+B) &\geq \min(w_k(A), w_k(B)) \\ w_k(AB) &\geq \min_{i+j \leq k} (w_i(A) + w_j(B)). \end{aligned}$$

4.2 Log decay period rings

Our period rings will be subrings of $\tilde{\mathbf{B}}$ with growth conditions on the valuations of the Teichmüller representatives. Informally $\tilde{\mathbf{B}}^r$ is the subset of $\tilde{\mathbf{B}}$ consisting of elements where the growth of $-v_T(x_n)$ is no faster than p^{rn} . Similarly, $\tilde{\mathbf{B}}^\dagger$ is the subset of $\tilde{\mathbf{B}}$ where the growth of $v_T(x_n)$ is bounded by some linear function (i.e. the overconvergent elements). More precisely

$$\tilde{\mathbf{B}}^r := \left\{ \sum_{n \gg -\infty} [x_n] p^n \in \tilde{\mathbf{B}} \mid \begin{array}{l} \text{There exists } c > 0 \text{ such that} \\ v_T(x_n) \geq -p^{nr}c \end{array} \right\}$$

$$\tilde{\mathbf{A}}^{r,c} := \left\{ \sum_{n=0}^{\infty} [x_n] p^n \in \tilde{\mathbf{B}}^r \mid p^{nr}c + v_T(x_n) \geq c \right\}$$

$$\tilde{\mathbf{B}}^\dagger := \left\{ \sum_{n \gg -\infty} [x_n] p^n \in \tilde{\mathbf{B}} \mid \begin{array}{l} \text{There exists } c > 0 \text{ such that} \\ nc + v_T(x_n) \gg -\infty \text{ as } n \rightarrow \infty \end{array} \right\}.$$

In addition we define $\tilde{\mathbf{A}}^r = \tilde{\mathbf{A}} \cap \tilde{\mathbf{B}}^r$ and $\tilde{\mathbf{A}}^\dagger = \tilde{\mathbf{A}} \cap \tilde{\mathbf{B}}^\dagger$. We remark that $\tilde{\mathbf{A}}^{r,c} \subset \tilde{\mathbf{A}}^r$. It is not obvious a priori that these sets are subrings of $\tilde{\mathbf{B}}$.

Lemma 4.1. *The sets $\tilde{\mathbf{B}}^r$, $\tilde{\mathbf{A}}^r$, $\tilde{\mathbf{A}}^{r,c}$, $\tilde{\mathbf{B}}^\dagger$, and $\tilde{\mathbf{A}}^\dagger$ are subrings of $\tilde{\mathbf{B}}$.*

Proof. The sets $\tilde{\mathbf{B}}^\dagger$ and $\tilde{\mathbf{A}}^\dagger$ are commonplace in p -adic Hodge theory and are known to be rings. The rest of the assertion can be proven from the inequalities (1) and (2). We provide a proof for $\tilde{\mathbf{A}}^{r,c}$ as an example. Let

$$x = \sum_{n=0}^{\infty} [x_n] p^n \quad \text{and} \quad y = \sum_{n=0}^{\infty} [y_n] p^n$$

be two elements of $\tilde{\mathbf{A}}^{r,c}$. Inequality (1) immediately gives $x + y \in \tilde{\mathbf{A}}^{r,c}$ so we need to show, so we need to show that $xy \in \tilde{\mathbf{A}}^{r,c}$. Our definition of $\tilde{\mathbf{A}}^{r,c}$ implies that $w_n(x), w_n(y) \geq c - p^{nr}c$ for all n . Let $i, j \geq 0$ be whole numbers such that $i + j \geq n$ and $w_n(xy) \geq w_i(x) + w_j(y)$. Note that $p^{nr} + 1 \geq p^{ir} + p^{jr}$. This together with inequality (2) gives

$$\begin{aligned} w_n(xy) &\geq w_i(x) + w_j(y) \\ &\geq 2c - c(p^{ir} + p^{jr}) \\ &\geq 2c - c(p^{nr} + 1) \\ &= c - p^{nr}c, \end{aligned}$$

which finishes our claim. \square

Lemma 4.2. *The Frobenius map \mathbf{Frob} on $\tilde{\mathbf{B}}$ induces isomorphism on $\tilde{\mathbf{B}}^\dagger, \tilde{\mathbf{A}}^\dagger, \tilde{\mathbf{B}}^r$, and $\tilde{\mathbf{A}}^r$. Furthermore, it induces an isomorphism*

$$\mathbf{Frob} : \tilde{\mathbf{A}}^{r,c} \xrightarrow{\sim} \tilde{\mathbf{A}}^{r,pc}.$$

Proof. Let $x = \sum_{n=0}^{\infty} [x_n]p^n \in \tilde{\mathbf{A}}^{r,c}$. Then $\mathbf{Frob}(x) = \sum_{n=0}^{\infty} [x_n^p]p^n$. We find

$$\begin{aligned} v_T(x_n^p) - (cp)p^{nr} &= p(v_T(x_n) - cp^{nr}) \\ &\geq pc, \end{aligned}$$

so that $\mathbf{Frob} : \tilde{\mathbf{A}}^{r,c} \hookrightarrow \tilde{\mathbf{A}}^{r,pc}$. As F^{alg} is perfect we may consider \mathbf{Frob}^{-1} and similar inequalities show $\mathbf{Frob}^{-1} : \tilde{\mathbf{A}}^{r,pc} \hookrightarrow \tilde{\mathbf{A}}^{r,c}$. This proves the claim for $\tilde{\mathbf{A}}^{r,c}$. The proof for the other rings are similar and are standard in the overconvergent case (see [3, I.3]). \square

Theorem 4.3. *Let σ be a Frobenius endomorphism of \mathcal{E}^r . Then there exists a unique embedding*

$$i_\sigma : \mathcal{E}^r \hookrightarrow \tilde{\mathbf{B}}^r$$

satisfying:

- i_σ is a morphism of K -algebras
- i_σ induces the inclusion $F \hookrightarrow F^{alg}$ on residue fields
- i_σ commutes with Frobenius (i.e. $i_\sigma \circ \sigma = \mathbf{Frob} \circ i_\sigma$)

Proof. The existence and uniqueness of $i_\sigma : \mathcal{E}^r \hookrightarrow \tilde{\mathbf{B}}$ satisfying the desired properties is described by Tsuzuki (see [24, Lemma 2.5.1]). Therefore it suffices to prove that the image of i_σ lies in $\tilde{\mathbf{B}}^r$. Let T be a local parameter of \mathcal{E}^r , so that $\mathcal{E}^r = \mathcal{E}_{K,T}^r$, and let $P(T) = \sigma(T)$. Then $v = i_\sigma(T)$ satisfies the equation $\mathbf{Frob}(x) = P(x)$. The theorem will follow from the following Lemma. \square

Lemma 4.4. *Let $v \in \tilde{\mathbf{A}}$ be a solution of $\mathbf{Frob}(x) = P(x)$ with $w_0(v) = 1$. Then $v \in \tilde{\mathbf{A}}^r$.*

Proof. Write $P(T) = \sum_{n=-\infty}^{\infty} a_n T^n$. Note that $a_p = 1$ and that $p|a_n$ whenever $n \neq p$. There exists $c \in \mathbb{R}$ such that

$$v_p(a_n) - \frac{\log_p(-n)}{r} > c,$$

for all $n < 0$. Choose d to be large enough so that $d > \frac{1}{p^r-1}$, $pd > \frac{1}{p^{rc}-1}$, and $pd > \frac{p^{r-cr}}{p^r-1}$.

We claim that for any $x \in \tilde{\mathbf{A}}^{r,d} + p^k \tilde{\mathbf{A}}$ with $w_0(x) = 1$ we have $P(x) \in \tilde{\mathbf{A}}^{r,pd} + p^{k+1} \tilde{\mathbf{A}}$. To do this, it is enough to show that $a_n x^n \in \tilde{\mathbf{A}}^{r,pd} + p^{k+1} \tilde{\mathbf{A}}$ for each n . When $n = p$ we find that $x^p \in \tilde{\mathbf{A}}^{r,pd} + p^{k+1} \tilde{\mathbf{A}}$ by checking the binomial expansion. When $n \geq 0$ and $n \neq p$ we have $p|a_n$, so we must have $a_n x^n \in \tilde{\mathbf{A}}^{r,pd} + p^{k+1} \tilde{\mathbf{A}}$. The difficult part is dealing with negative exponents of x .

Write $x = [x_0] + [x_1]p + \dots + [x_{k-1}]p^{k-1} + x'p^k$. For $n < 0$ we know that $e = \max(c + \frac{\log_p(-n)}{r}, 1)$ is less than or equal to $v_p(a_n)$. In particular if $p^e x^n$ is contained in $\tilde{\mathbf{A}}^{r,pd} + p^{k+1} \tilde{\mathbf{A}}$ then so is $a_n x^n$. We find that

$$x^{-1} = [x_0] \frac{1}{1 + [\frac{x_1}{x_0}]p + \dots + [\frac{x_{k-1}}{x_0}]p^{k-1} + x'p^k}.$$

Expanding this geometric sum we see that $p^e x^n$ consists of terms of the form

$$p^e [x_0^n] \prod_{i=1}^k \left(\left[\frac{x_i}{x_0} \right] p^i \right)^{n_i}.$$

Thus it is enough to show that $[\frac{x_i}{x_0}]p^i$ and $p^e [x_0^n]$ are contained in $\tilde{\mathbf{A}}^{r,pd}$.

To prove that $[\frac{x_i}{x_0}]p^i \in \tilde{\mathbf{A}}^{r,pd}$ it suffices to show

$$v_T\left(\frac{x_i}{x_0}\right) = v_T(x_i) - 1 > pd - p^{ir}pd. \quad (\text{A})$$

As $d > \frac{1}{p^r-1}$ we see that $d > \frac{1}{p^{ir}-1}$ and therefore

$$(p-1)(p^{ir}-1)d > 1. \quad (B)$$

Subtracting (B) from the inequality $v_T(x_i) > d - p^{ir}d$ yields (A). To show $p^e[x_0^n] \in \tilde{\mathbf{A}}^{r,pd}$ we must prove the inequality

$$v_T(x_0^n) = n > pd - p^{re}pd. \quad (C)$$

When $c + \frac{\log_p(-n)}{r} \geq 1$, our definition of e allows us to rewrite (C) as

$$n - p^{rc}npd > pd. \quad (C1)$$

From $pd > \frac{1}{p^{rc}-1}$ we see that $p^{rc}pd - 1 > pd$. Since $-n$ is positive we then have $-n(p^{rc}pd - 1) > pd$, which gives (C1). If $c + \frac{\log_p(-n)}{r} < 1$, then $e = 1$ and we can rewrite (C) as

$$n + p^rpd > pd. \quad (C2)$$

As $pd > \frac{p^{r-cr}}{p^r-1}$ we see that

$$-p^{r-cr} + p^rpd > pd. \quad (D)$$

The condition $c + \frac{\log_p(-n)}{r} < 1$ implies $n > -p^{r-cr}$, which combines with (D) to give (C2). This concludes the proof of $P(x) \in \tilde{\mathbf{A}}^{r,pd} + p^{k+1}\tilde{\mathbf{A}}$.

We know that $v_T(v_0) = 1$ and $v \in \tilde{\mathbf{A}}^{r,d} + p\tilde{\mathbf{A}}$. Proceeding inductively, we will assume that $v \in \tilde{\mathbf{A}}^{r,d} + p^k\tilde{\mathbf{A}}$. By the paragraphs above we have $\mathbf{Frob}(v) = P(v) \in \tilde{\mathbf{A}}^{r,pd} + p^{k+1}\tilde{\mathbf{A}}$. Then Lemma 4.2 tells us that $v \in \tilde{\mathbf{A}}^{r,d} + p^{k+1}\tilde{\mathbf{A}}$. It follows that $v \in \tilde{\mathbf{A}}^{r,d} \subset \tilde{\mathbf{B}}^r$. \square

Let σ be a Frobenius of \mathcal{E}^r . We may view \mathcal{E}^r and \mathcal{E} as subrings of $\tilde{\mathbf{B}}$ through i_σ . By Theorem 4.3 we know that \mathcal{E}^r is actually a subring of $\tilde{\mathbf{B}}^r$. A natural question is whether $\tilde{\mathbf{B}}^r$ contains any element of \mathcal{E} that does not have r -log-decay. More succinctly: is $\mathcal{E} \cap \tilde{\mathbf{B}}^r$ equal to \mathcal{E}^r ? This answer is affirmative by Corollary 4.7. To prove this fact we introduce an auxiliary subring of $\mathcal{O}_{\mathcal{E}^r}$ defined in an analogous manner as $\tilde{\mathbf{A}}^{r,c}$.

$$\mathcal{O}_{\mathcal{E}^r,c} := \left\{ f(T) \in \mathcal{O}_{\mathcal{E}} \mid v_n^{\text{naive}}(f(T)) \geq c - p^{nr}c \right\}$$

Lemma 4.5. *Let $c > 0$ be large enough so that $T[T^{-1}] \in \tilde{\mathbf{A}}^{r,c}$. Then*

$$w_n(T^k) \geq c - p^{rn}c + k$$

for all k .

Proof. Let $\alpha = T[T^{-1}]$. First we prove that $w_n(\alpha^k) \geq c - p^{rn}c$ for any $k \geq 1$ by induction on k . When $k = 1$ the inequality is true because $\alpha \in \tilde{\mathbf{A}}^{r,c}$. For $k > 1$ we have the inequality

$$w_n(\alpha^k) \geq \min_{i+j=n} \{w_i(\alpha^{k-1}) + w_j(\alpha)\}.$$

When $i = n$ and $j = 0$ the term in the minimum is greater than $c - p^{rn}c$ by our inductive hypothesis and since $w_0(\alpha) = 0$. Similarly for $i = 0$ and $j = n$. For $0 < i, j < n$ the term in the minimum is greater than $2c - p^{ri}c - p^{rj}c$, which is greater than $c - p^{rn}c$. It follows that $w_n(\alpha^k) \geq c - p^{rn}c$. For negative exponents, we will prove that $w_n(\alpha^{-k}) \geq c - p^{rn}c$ by induction on n . For $n = 0$ we know that $w_0(\alpha^{-k}) = 0$. Now consider the inequality

$$0 = w_n(\alpha^k \alpha^{-k}) \geq \min_{i+j=n} \{w_i(\alpha^k) + w_j(\alpha^{-k})\}.$$

Using our inductive hypothesis we know that for $i > 0$ each term in the minimum is greater than or equal to $c - p^{rn}c$. For $i = 0$ we know that $w_0(\alpha^k) = 0$. Therefore if $w_n(\alpha^{-k})$ is less than

$c - p^{rn}c$, the minimum is less than $c - p^{rn}c$. This minimum is achieved exactly once, meaning the inequality is actually an equality. This gives a contradiction. We now see that for any k we have

$$w_n(T^k) = w_n(\alpha^k[T^k]) = w_n(\alpha^k) + k,$$

which gives the desired inequality. □

Proposition 4.6. *Let $c > 0$ be large enough so that $T[T^{-1}] \in \tilde{\mathbf{A}}^{r,c}$. Then*

$$\tilde{\mathbf{A}}^{r,c} \cap \mathcal{O}_{\mathcal{E}} = \mathcal{O}_{\mathcal{E}^{r,c}}.$$

Proof. Let $f(T) \in \tilde{\mathbf{A}}^{r,c} \cap \mathcal{O}_{\mathcal{E}}$ and let $a_n = v_n^{\text{naive}}(f(T))$. We may write

$$f(T) = \sum_{n=0}^{\infty} q_n(T)p^n,$$

where $q_n(T) = u_n(T)T^{a_n}$ and $u_n(T)$ is a unit in $\mathcal{O}_K[[T]]$. Define the partial p -adic sums

$$s_n(T) = \sum_{i=0}^n q_i(T)p^i.$$

By induction on n we will simultaneously prove that $a_n \geq c - p^{rn}c$ and that the partial sum $s_n(T)$ is contained in $\tilde{\mathbf{A}}^{r,c}$. When $n = 0$ there is nothing to check. For $n > 0$ we have

$$w_{n+1}(s_{n+1}(T)) = w_{n+1}(f(T)) \geq c - p^{r(n+1)}c,$$

since $f(T) \in \tilde{\mathbf{A}}^{r,c}$. Also, by our inductive hypothesis we know that $s_n(T) \in \tilde{\mathbf{A}}^{r,c}$, which means

$$w_{n+1}(s_n(T)) \geq c - p^{r(n+1)}c.$$

By the inequality (1) we know

$$w_{n+1}(s_{n+1}(T)) \geq \min\{w_{n+1}(s_n(T)), w_{n+1}(q_{n+1}(T)p^{n+1})\}.$$

If $w_{n+1}(q_{n+1}(T)p^{n+1}) < c - p^{r(n+1)}c$ the above is an equality, which yields a contradiction. Since $a_{n+1} = w_0(q_{n+1}(T)) = w_{n+1}(q_{n+1}(T)p^{n+1})$ we see that $a_{n+1} \geq c - p^{r(n+1)}c$.

It remains to prove $s_{n+1}(T) \in \tilde{\mathbf{A}}^{r,c}$. Since $s_n(T)$ and $u_n(T)$ are both contained in $\tilde{\mathbf{A}}^{r,c}$ it is enough to prove $T^{a_{n+1}}p^{n+1} \in \tilde{\mathbf{A}}^{r,c}$. For $k < n+1$ we have $w_k(T^{a_{n+1}}p^{n+1}) = \infty$. By Lemma 4.5 we have for $k \geq 0$

$$\begin{aligned} w_{n+1+k}(T^{a_{n+1}}p^{n+1}) &= w_k(T^{a_{n+1}}) \\ &\geq c - p^{rk}c + a_{n+1} \\ &\geq c - p^{rk}c + c - p^{r(n+1)}c \\ &\geq c - p^{r(n+1+k)}c, \end{aligned}$$

which gives $T^{a_{n+1}}p^{n+1} \in \tilde{\mathbf{A}}^{r,c}$. It follows that $\tilde{\mathbf{A}}^{r,c} \cap \mathcal{O}_{\mathcal{E}} \subset \mathcal{O}_{\mathcal{E}^{r,c}}$.

Conversely, let $f(T)$ be in $\mathcal{O}_{\mathcal{E}^{r,c}}$ and let a_n be $v_n^{\text{naive}}(f(T))$. We need to show $f(T) \in \tilde{\mathbf{A}}^{r,c}$. Just as before we may write

$$f(T) = \sum_{n=0}^{\infty} q_n(T)p^n,$$

where $q_n(T) = u_n(T)T^{a_n}$ and $u_n(T)$ is a unit in $\mathcal{O}_K[[T]]$. Since $\tilde{\mathbf{A}}^{r,c}$ is complete with respect to the p -adic topology, it will suffice to prove that $q_n(T)p^n \in \tilde{\mathbf{A}}^{r,c}$ for each n . We also know that

$u_n(T) \in \tilde{\mathbf{A}}^{r,c}$, so we just need to show $T^{a_n}p^n \in \tilde{\mathbf{A}}^{r,c}$. Since $a_n \geq c - p^{rn}c$ we see from Lemma 4.5

$$\begin{aligned} w_{k+n}(T^{a_n}p^n) &= w_k(T^{a_n}) \\ &\geq c - p^{rk}c + c - p^{rn}c \\ &\geq c - p^{r(k+n)}c. \end{aligned}$$

□

Corollary 4.7. *Let $c > 0$ be large enough so that $T[T^{-1}] \in \tilde{\mathbf{A}}^{r,c}$. Then*

$$\tilde{\mathbf{B}}^r \cap \mathcal{E} = \mathcal{E}^r.$$

Proof. This follows from Proposition 4.6 combined with the fact that $\tilde{\mathbf{A}}^{r,c}[\frac{1}{p}] = \tilde{\mathbf{B}}^r$ and $\mathcal{O}_{\mathcal{E}^{r,c}}[\frac{1}{p}] = \mathcal{E}^r$. □

Let $\mathcal{E}^{r'}$ be an unramified extension of \mathcal{E}^r with a Frobenius endomorphism σ extending that on \mathcal{E}^r . By Theorem 4.3 we may extend i_σ to an embedding of $\mathcal{E}^{r'}$ in $\tilde{\mathbf{B}}^r$. In particular, we may embed the maximal unramified extension of \mathcal{E}^r into $\tilde{\mathbf{B}}^r$ in a Frobenius compatible way. We let $\tilde{\mathcal{E}}^r$ denote the p -adic completion of the maximal extension of \mathcal{E}^r inside of $\tilde{\mathbf{B}}^r$ and we let $\mathcal{O}_{\tilde{\mathcal{E}}^r} = \tilde{\mathbf{A}} \cap \tilde{\mathcal{E}}^r$. Similarly we define $\tilde{\mathcal{E}}$ to be the p -adic completion of the maximal unramified extension of \mathcal{E} in $\tilde{\mathbf{B}}$ and we let $\mathcal{O}_{\tilde{\mathcal{E}}} = \tilde{\mathbf{A}} \cap \tilde{\mathcal{E}}$. Note that $\tilde{\mathcal{E}}^r \subset \tilde{\mathcal{E}}$. There is an action of G_F on $\tilde{\mathcal{E}}^r$, $\mathcal{O}_{\tilde{\mathcal{E}}^r}$, $\tilde{\mathcal{E}}$, and $\mathcal{O}_{\tilde{\mathcal{E}}}$. The invariants are $(\tilde{\mathcal{E}}^r)^{G_F} = \mathcal{E}^r$, $\mathcal{O}_{\tilde{\mathcal{E}}^r}^{G_F} = \mathcal{O}_{\mathcal{E}^r}$, $\tilde{\mathcal{E}}^{G_F} = \mathcal{E}$ and $\mathcal{O}_{\tilde{\mathcal{E}}}^{G_F} = \mathcal{O}_{\mathcal{E}}$. For a finite separable extension K over F we have $\mathcal{O}_{\mathcal{E}_K^{r,c}} = \mathcal{E}_K \cap \tilde{\mathbf{A}}^{r,c}$ by Proposition 4.6. We define $\mathcal{O}_{\tilde{\mathcal{E}}^{r,c}} = \tilde{\mathcal{E}} \cap \tilde{\mathbf{A}}^{r,c}$. Then $\mathcal{O}_{\tilde{\mathcal{E}}^{r,c}}$ is the p -adic completion of the $\cup_{K/F} \mathcal{O}_{\mathcal{E}_K^{r,c}}$, where the union is taken over all finite separable extensions K over F .

Remark 4.8. *The analogous result to Theorem 4.3 holds for overconvergent Frobenius endomorphisms. This allows us to define a period ring \mathbf{B}^\dagger , which is the p -adic completion of the maximal unramified extension of \mathcal{E}^\dagger in $\tilde{\mathbf{B}}$. When $\sigma(T) = (1+T)^p - 1$ then \mathbf{B}^\dagger is the same period ring that arises in p -adic Hodge theory. The ring \mathbf{B}^\dagger is strictly bigger than the period ring $\tilde{\mathcal{E}}^\dagger$ used by Tsuzuki. The ring $\tilde{\mathcal{E}}^\dagger$ is the maximal unramified extension of \mathcal{E}^\dagger . The ring \mathbf{B}^\dagger contains elements transcendental over \mathcal{E}^\dagger , while $\tilde{\mathcal{E}}^\dagger$ consists only of algebraic elements. Tsuzuki proves that all (ϕ, ∇) -modules over \mathcal{E}^\dagger become trivial after a finite extension (i.e. they are quasi-constant), which means only algebraic elements are needed. In contrast, the (ϕ, ∇) -modules we will consider only trivialize after adding transcendental periods.*

4.3 Some period modules

In the proof of Theorem 1.2 we will need to utilize some *period submodules* of $\tilde{\mathcal{E}}^r$ with very precise growth properties. Recall from the proof of Theorem 4.3 that there exists $c_0 > 0$ such that $T \in \tilde{\mathbf{A}}^{r,c_0}$. For $c > c_0$ and $n > 0$ we define

$$\tilde{\mathcal{E}}^{r,c,n} := \left\{ \sum_{i=0}^{\infty} [x_i]p^i \in \tilde{\mathcal{E}} \mid \begin{array}{l} p^{ir}c + v_T(x_i) \geq c \quad \text{for } i < n \\ p^{(n-1)r}c + |w_1(T)| + v_T(x_n) \geq c \end{array} \right\}.$$

Consider an element $\sum [x_i]p^i$ of $\tilde{\mathcal{E}}^{r,c,n}$. The first n terms ($i < n$) will look like an arbitrary element of $\mathcal{O}_{\tilde{\mathcal{E}}^{r,c}}$ (i.e. $\tilde{\mathcal{E}}^{r,c,n}/p^n = \mathcal{O}_{\tilde{\mathcal{E}}^{r,c}}/p^n$). For $i > n+1$ the valuations of x_i have no restrictions. However the bound on $v_T(x_n)$ is essentially the same as the bound on $v_T(x_{n-1})$ up to a constant that does not depend on n . The slight difference in these bounds is to allow T to act on $\tilde{\mathcal{E}}^{r,c,n}$ via multiplication.

Lemma 4.9. *Let $c > c_0 \frac{p^{2r}-1}{p^r-1}$. Then multiplication by T sends $\tilde{\mathcal{E}}^{r,c,n}$ to $\tilde{\mathcal{E}}^{r,c,n}$*

Proof. Let $x = \sum [x_i]p^i \in \tilde{\mathcal{E}}^{r,c,n}$ and write $y = Tx = \sum [y_i]p^i$. Since T and x are contained in $\mathcal{O}_{\tilde{\mathcal{E}}^{r,c}} + p^n \mathcal{O}_{\tilde{\mathcal{E}}}$ we know $y \in \mathcal{O}_{\tilde{\mathcal{E}}^{r,c}} + p^n \mathcal{O}_{\tilde{\mathcal{E}}}$. This leaves us with checking $v_T(y_n) \geq c - p^{(n-1)r}c + w_1(T)$. By inequality (2) we have

$$v_T(y_n) \geq \min_{i+j=n} \{w_i(x) + w_j(T)\}$$

so it will suffice to prove

$$w_i(x) + w_j(T) \geq c - p^{(n-1)r}c - w_1(T),$$

whenever $i + j = n$. We have $w_n(x) \geq c - p^{(n-1)r}c + w_1(T)$ since $x \in \tilde{\mathcal{E}}^{r,c,n}$ and $w_0(T) = 1$, which proves the inequality for $i = n$ and $j = 0$. Similarly we have $w_{n-1}(x) \geq c - p^{(n-1)r}c$, which gives the inequality for $i = n-1$ and $j = 1$. Finally we assume that $1 < i, j < n-1$. The inequality in the hypothesis gives

$$c_0 - p^{jr}c_0 > c - p^{(j-1)r}c.$$

Then since $w_i(x) \geq c - p^{ir}c$ and $T \in \tilde{\mathbf{A}}^{r,c_0}$ we have

$$\begin{aligned} w_i(x) + w_j(T) &\geq c - p^{ir}c + c - p^{(j-1)r}c \\ &\geq c - p^{nr}c. \end{aligned}$$

□

For a finite extension K over F we define $\mathcal{E}_K^{r,c,n} = \tilde{\mathcal{E}}^{r,c,n} \cap (\mathcal{O}_{\mathcal{E}_K} + p^{n+1}\mathcal{O}_{\mathcal{E}})$. Informally we may think of $\mathcal{E}_K^{r,c,n}$ as elements in $\tilde{\mathcal{E}}^{r,c,n}$ that look like elements of $\mathcal{O}_{\mathcal{E}_K}$ when reduced modulo p^{n+1} . In particular, there is an action of $G_{K/F}$ on $\mathcal{E}_K^{r,c,n}/p^{n+1}$.

4.4 The functors $D_{\mathcal{E}}$, $D_{\mathcal{E}^\dagger}$, and $D_{\mathcal{E}^r}$

Let R be one of \mathcal{E} , \mathcal{E}^\dagger , or \mathcal{E}^r all with residue field F . Fix a Frobenius σ on R . Denote by $\mathbf{Rep}_{\mathbb{Q}_p}(G_F)$ the category of continuous \mathbb{Q}_p representations of $\text{Gal}(F^{sep}/F)$ and let $\mathbf{Rep}_{\mathbb{Q}_p}^{fin}(G_F)$ be the subcategory of representations where the image of the inertia group is finite. In this section we will define a functor D_R from the $\mathbf{Rep}_{\mathbb{Q}_p}(G_F)$ to $\mathbf{M}\Phi_{R,\sigma}^\nabla$ and we describe results of Fontaine and Tsuzuki about these functors.

Definition 4.10. Let $V \in \mathbf{Rep}_{\mathbb{Q}_p}(G_F)$. Define $D_R(V)$ to be $(\tilde{R} \otimes V)^{G_F}$. The connection on $D_R(V)$ is given by $\nabla = \delta_T \otimes 1$ and the Frobenius semi-linear morphism is given by $\phi = \sigma \otimes 1$. We say that $V \in \mathbf{Rep}_{\mathbb{Q}_p}(G_F)$ is \tilde{R} -admissible if $\dim_K V = \dim_R D_R(V)$.

Theorem 4.11. The functor

$$D_{\mathcal{E}} : \mathbf{Rep}_{\mathbb{Q}_p}(G_F) \rightarrow \mathbf{M}\Phi_{\mathcal{E},\sigma}^{et,\nabla}$$

is an equivalence of categories. The functor

$$D_{\mathcal{E}^\dagger} : \mathbf{Rep}_{\mathbb{Q}_p}^{fin}(G_F) \rightarrow \mathbf{M}\Phi_{\mathcal{E}^\dagger,\sigma}^{et,\nabla}$$

is an equivalence of categories. In other words, all Galois representations are $\tilde{\mathcal{E}}$ -admissible and then $\tilde{\mathcal{E}}^\dagger$ -admissible Galois representations are those with finite monodromy.

Proof. The first statement is due to Fontaine ([10]) and the second statement is due to Tsuzuki ([24]). □

5 Ramification theory

5.1 The higher ramification groups

We first recall the definition and basic properties of the higher ramification groups. Let L be a separable extension of $F = k((T))$ such that $G_{L/F}$ is a finite dimensional p -adic Lie group. Let v denote the T -adic valuation on F normalized so that $v(T) = 1$. When L is a finite extension we let T_L denote a uniformizing element of L and we let v_L denote the valuation on L normalized so that $v_L(T_L) = 1$. For $s \in \mathbb{R}_{\geq 0}$ there is an upper numbering ramification group $G_{L/F}^s$ satisfying:

- When $t > s$ we have $G_{L/F}^t \subset G_{L/F}^s$.
- The group $G_{L/F}^0$ is equal to $I_{L/F}$, the inertia group of $G_{L/F}$.
- The intersection

$$\bigcap_{s \geq 0} (G_{L/F})^s = \{0\}$$

- Let K be a finite extension of F contained in L . Then $G_{K/F}^s = \frac{G_{L/F}^s G_{L/K}}{G_{L/K}}$.

Define the function

$$\psi_{L/F}(y) = \int_0^y [G_{L/F}^0 : G_{L/F}^s] ds.$$

Since the function $\psi_{L/F}$ is monotone increasing there is an inverse function $\phi_{L/F}$. The *ramification polygon* of L over K is the graph of $y = \phi_{L/F}(x)$. When L is a finite extension of F there are lower ramification groups satisfying $(G_{L/F})_x = (G_{L/F})^{\phi_{L/F}(x)}$. These lower ramification groups may be defined explicitly as follows:

$$(G_{L/F})_x = \{g \in G_{L/F} \mid v_L(g(T_L) - T_L) \geq x\}.$$

In the case where $[L : F]$ is finite this filtration is finite. That is, for large enough s the groups $G_{L/F}^s$ and $(G_{L/F})_s$ contain only the identity element. We let $\lambda_{L/F}$ (resp. $\mu_{L/K}$) denote the largest number such that $(G_{L/F})_s$ (resp. $G_{L/F}^s$) is not trivial.

5.2 Some auxillary results on the higher ramification groups

In the remainder of this section we will prove several auxiliary lemmas about the higher ramification groups that will be used in the proof of Theorem 1.2.

Lemma 5.1. *Let L be a totally ramified finite extension of F and let $s > 0$ satisfy $G_{L/F}^s = 0$. Then for any $g \in G_{L/F}$ we have $v(T_L - g(T_L)) < s$.*

Proof. Let (x_0, y_0) be the coordinates for the last break in the ramification polygon of $G_{L/F}$. We remark that x_0 is the largest number such that $(G_{L/F})_{x_0}$ is not trivial. Since $G_{L/F}^s = 0$ we know that $y_0 < s$. Let $m = |G_{L/F}|$. At any x we know that $\phi_L'(x) \geq \frac{1}{m}$, provided the derivative exists, with equality for $x > x_0$. This implies $y_0 \geq \frac{x_0}{m}$, which means that $(G_{L/F})_{y_0 m} = (G_{L/F})_{sm} = 0$. Thus $v_L(T_L - g(T_L)) < sm$ for all g . The inertia degree of L over F is m , so that $mv = v_L$, which proves the Lemma. \square

Lemma 5.2. *Let L be a totally ramified finite extension of F . Then*

$$v(x - g(x)) \geq v(x) + \frac{\lambda_{L/F} - 1}{|G_{L/F}|}$$

for any $x \in L^\times$ and $g \in (G_{L/F})_{\lambda_{L/F}}$.

Proof. Let $x, y \in L$ such that $v(x) \neq v(y)$. If the Lemma holds for both x and y , then the inequality also holds for $x + y$. Also, if the Lemma holds for cx when $c \in F$, then the Lemma holds for x . Consider the decomposition of L :

$$\bigoplus_{i=0}^{[L:F]-1} T_L^i F.$$

When $x \in T_L^i F$ and $y \in T_L^j$ with $i \neq j$, we have $v(x) \neq v(y)$, so by the above remarks it will suffice to prove the statement for powers of T_L . We have

$$T_L^i - g(T_L^i) = (T_L - g(T_L))(T_L^{i-1} + \dots + g(T_L)^{i-1}).$$

Then $v(T_L - g(T_L)) = \frac{\lambda_{L/F}}{|G_{L/F}|}$ and each term in the sum term

$$T_L^{i-1} + \dots + g(T_L)^{i-1}$$

has T -adic valuation $v(T_L^i) - \frac{1}{|G_{L/F}|}$, which implies the lemma. \square

Lemma 5.3. *Let K be a finite extension of F and let L be a finite extension of K . Then*

$$\mu_{L/F} \leq \frac{\lambda_{L/F}}{|G_{K/F}|} + \mu_{K/F}.$$

Proof. By definition we have $\phi_{K/F}(\lambda_{K/F}) = \mu_{K/F}$. For any $s > \lambda_{K/F}$ we have $\phi'_{K/F}(s) = \frac{1}{|G_{K/F}|}$ and therefore $\phi_{K/F}(s) = \mu_{K/F} + \frac{s - \lambda_{K/F}}{|G_{K/F}|}$. A general property of ramification polygons is that $\phi_{L/F}(s) \leq \phi_{K/F}(s)$ for all s . In particular if we have $\lambda_{L/F} \geq \lambda_{K/F}$ then

$$\begin{aligned} \mu_{L/F} &= \phi_{L/F}(\lambda_{L/F}) \\ &\leq \phi_{K/F}(\lambda_{L/F}) \\ &= \mu_{K/F} + \frac{\lambda_{L/F} - \lambda_{K/F}}{|G_{K/F}|} \\ &\leq \frac{\lambda_{L/F}}{|G_{K/F}|} + \mu_{K/F}. \end{aligned}$$

If $\lambda_{L/F} < \lambda_{K/F}$ then

$$\begin{aligned} \mu_{L/F} &= \phi_{L/F}(\lambda_{L/F}) \\ &\leq \phi_{K/F}(\lambda_{L/F}) \\ &< \phi_{K/F}(\lambda_{K/F}) \\ &= \mu_{K/F}. \end{aligned}$$

\square

Lemma 5.4. *Let K be an extension of F and let L be an extension of K of degree p such that $(G_{L/K})_s = 0$. Then $H^1(G_{L/K}, \mathcal{O}_L)$ is killed by $T^{\lceil \frac{s}{p} \rceil}$.*

Proof. The ramification filtration of $G_{L/K}$ has one break. By the Hasse-Arf theorem the upper numbering of this break is an integer n . The lower numbering of the break is n and we have $n < s$. That is, $v_L(T_L - g_0(T_L)) = n$, where g_0 is a generator of $G_{L/K}$.

Consider $N = \sum_{g \in G_{L/K}} g$ and $r = 1 - g_0$, viewed as elements of $\mathbb{Z}[G_{L/K}]$. Both N and r define \mathcal{O}_K -linear morphisms from \mathcal{O}_L to itself. There is an isomorphism (see [2, Section 8]):

$$H^1(G_{L/K}, \mathcal{O}_L) = \ker(N)/\text{Im}(r),$$

so it suffices to consider $\ker(N)/\text{Im}(r)$. The kernel of r is $\mathcal{O}_K \subset \mathcal{O}_L$ and \mathcal{O}_L has \mathcal{O}_K -rank p , which means that the rank of $\text{Im}(r)$ as an \mathcal{O}_K -module is $p - 1$. Similarly, the image of N is contained in \mathcal{O}_K , so that the rank of $\ker(N)$ as an \mathcal{O}_K -module is $p - 1$. Therefore $\ker(N)/\text{Im}(r)$ is a torsion \mathcal{O}_K -module.

We have the following decomposition

$$\mathcal{O}_L = \mathcal{O}_K \oplus T_L \mathcal{O}_K \oplus \dots T_L^{p-1} \mathcal{O}_K.$$

Then $\text{Im}(r)$ will be the \mathcal{O}_K -module of rank $p - 1$ generated by the elements $e_i = g(T_L^i) - T_L^i$. We claim that $v_L(e_i) = n + i - 1$. First write

$$e_i = (g(T_L) - T_L) \sum_{k+j=i-1} T_L^k g(T_L)^j.$$

Since $g(T_L) \equiv T_L \pmod{T_L^2}$ we know that each term in the sum is equivalent modulo T_L^i . In particular we find

$$\sum_{k+j=i-1} T_L^k g(T_L)^j \equiv (p-1) T_L^{i-1} \pmod{T_L^i},$$

whose T_L -adic valuation is $i - 1$ and therefore $v_L(e_i) = n + i - 1$.

It follows that e_i is contained in $T_L^n \mathcal{O}_L$ and not in $T_L^n T \mathcal{O}_L$. Now consider the p -dimensional k -vector space $T_L^n \mathcal{O}_L / T_L^n T \mathcal{O}_L$. Since the e_i all have distinct valuations we know that their images in $T_L^n \mathcal{O}_L / T_L^n T \mathcal{O}_L$ are linearly independent. By Nakayama's Lemma there is $e' \in T_L^n \mathcal{O}_L$ such that e_1, \dots, e_{p-1}, e' is a \mathcal{O}_K -basis of $T_L^n \mathcal{O}_L$. Any $x \in \ker(N)$ can be written uniquely as a linear combination of e_1, \dots, e_{p-1} over K , since $\ker(N)/\text{Im}(r)$ is torsion. We know that $T^{\lceil \frac{n}{p} \rceil} x$ is in $T_L^n \mathcal{O}_L$ so it can be written uniquely as a linear combination of e_1, \dots, e_{p-1}, e' over \mathcal{O}_K . The coefficient of e' has to be zero. This means $T^{\lceil \frac{n}{p} \rceil} x$ is in $\text{Im}(r)$. It follows that $\ker(N)/\text{Im}(r)$ is killed by $T^{\lceil \frac{n}{p} \rceil} x$ and since $n < s$ the Lemma follows. \square

Lemma 5.5. *Let L be a finite extension of F and let s be an integer with $G_{L/F}^s = 0$. Then $H^1(G_{L/F}, \mathcal{O}_L)$ is killed by T^s .*

Proof. Let K be the maximal tamely ramified extension of F contained in L . The restriction-inflation sequence gives

$$0 \rightarrow H^1(G_{K/F}, \mathcal{O}_K) \rightarrow H^1(G_{L/F}, \mathcal{O}_L) \rightarrow H^1(G_{L/K}, \mathcal{O}_L) \rightarrow H^2(G_{K/F}, \mathcal{O}_K).$$

We know that \mathcal{O}_K is a projective $\mathcal{O}_F[G_{K/F}]$ -module, so $H^i(G_{K/F}, \mathcal{O}_K)$ is trivial for $i = 1, 2$. Therefore, it suffices to prove the theorem for totally wildly ramified extensions.

There is a tower of fields $L = L_{-1} \supset L_0 \supset \dots \supset L_r = F$ such that L_i is an extension of degree p over L_{i+1} . We will show $H^1(G_{L/L_i}, \mathcal{O}_L)$ is T^s -torsion by inducting on i . When $i = -1$ there is nothing to show. For $i \geq 0$, consider the restriction-inflation sequence

$$0 \rightarrow H^1(G_{L_{i-1}/L_i}, \mathcal{O}_{L_{i-1}}) \rightarrow H^1(G_{L/L_i}, \mathcal{O}_L) \rightarrow H^1(G_{L/L_{i-1}}, \mathcal{O}_L).$$

By our induction hypothesis $H^1(G_{L/L_{i-1}}, \mathcal{O}_L)$ is T^s -torsion, so it suffices to show that $H^1(G_{L_{i-1}/L_i}, \mathcal{O}_{L_{i-1}})$ is T^s -torsion. Let k be the integer such that $kp = |G_{L_{i-1}/F}|$. Since $G_{L_{i-1}/F}^s = 0$ we know from Lemma 5.1 that $v_{T_{L_{i-1}}}(T_{L_{i-1}} - g(T_{L_{i-1}})) < kps$ for $g \in G_{L_{i-1}/L_i}$. This means $(G_{L_{i-1}/L_i})_{kps} = 0$, so by Lemma 5.4 we know that $T_{L_i}^{ks}$ annihilates $H^1(G_{L_{i-1}/L_i}, \mathcal{O}_{L_{i-1}})$. The Lemma then follows from the fact that $v_T(T_{L_i}) = \frac{1}{k}$. \square

5.3 The different

In this subsection we will prove some auxiliary lemmas relating the different to higher ramification groups. These Lemmas will be used in Section 7 together with the Riemann-Hurwitz-Hasse formula to prove asymptotic results on genus growth (Theorem 7.2). Recall that for L a finite separable extension of F we define the different:

$$\delta_{L/F} = \sum_{i=0}^{\infty} (|G_{L/F}^i| - 1).$$

If $f(X)$ is the minimal polynomial of T_L then $\delta_{L/F} = v_L(f'(T_L))$.

Lemma 5.6. *Let L be a totally ramified finite extension of F . Then*

$$\frac{\delta_{L/F}}{|G_{L/F}|} = \mu_{L/F} - \frac{\lambda_{L/F}}{|G_{L/F}|}.$$

Proof. This is [21, Proposition IV.4]. □

Lemma 5.7. *With the same notation as Lemma 5.6, let $0 = s_0 < s_1 < \dots < s_n$ be real numbers with $s_n = \mu_{L/F}$. Then*

$$\lambda_{L/F} \leq \sum_{i=1}^n (s_i - s_{i-1}) [G_{L/F} : G_{L/F}^{s_i}].$$

Proof. This follows because $\lambda_{L/F} = \phi_{L/F}(\mu_{L/F})$. More precisely:

$$\begin{aligned} \lambda_{L/F} &= \int_0^{s_n} [G_{L/F} : G_{L/F}^s] ds \\ &= \sum_{i=1}^n \int_{s_{i-1}}^{s_i} [G_{L/F} : G_{L/F}^s] ds \\ &\leq \sum_{i=1}^n (s_i - s_{i-1}) [G_{L/F} : G_{L/F}^{s_i}]. \end{aligned}$$

□

Corollary 5.8. *Let L be a finite extension of F . Then*

$$\frac{\mu_{L/F}}{|G_{L/F}^{\mu_{L/F}}|} \leq \frac{\delta_{L/F}}{|G_{L/F}|}.$$

Proof. Combining Lemma 5.6 and Lemma 5.7 with $s_0 = 0 < s_1 = \mu_{L/F}$, we obtain

$$\begin{aligned} \frac{\delta_{L/F}}{|G_{L/F}|} &\geq \mu_{L/F} - (\mu_{L/F} - s_0) \frac{1}{|G_{L/F}^{\mu_{L/F}}|} \\ &\geq \mu_{L/F} \left(1 - \frac{1}{|G_{L/F}^{\mu_{L/F}}|} \right) \\ &\geq \frac{\mu_{L/F}}{|G_{L/F}^{\mu_{L/F}}|}, \end{aligned}$$

where the last inequality comes from the fact that $|G_{L/F}^{\mu_{L/F}}| \geq p$. □

6 Proof of Theorem 1.2

In this section we will prove Theorem 1.2. Let F be a local field of equal characteristic. In particular $F = k((T))$, where k is a finite field. Just as before we let v be the valuation on F normalized so that $v(T) = 1$. Let V be a d dimensional \mathbb{Q}_p -vector space and let $\rho : G_F \rightarrow GL(V)$ be a continuous representation. Fix a G_F -stable \mathbb{Z}_p -lattice $L \subset V$ of rank d . We let G be the image of ρ . The lattice L defines a p -adic Lie filtration on G :

$$G(n) = \ker(G \rightarrow GL(L/p^n L)).$$

We let F_n be the fixed field of $G(n)$ and let $H(n) = G/G(n)$. Note that $H(n)$ is the Galois group of F_n over F . Let μ_n (resp. λ_n) denote the largest ramification break with the upper (resp. lower) numbering of $H(n)$.

The proof of Theorem 1.2 will be broken up into three smaller propositions, which fit together as follows: Let $\mathbf{e} = (e_1, \dots, e_d)$ be a basis of L and let $\mathbf{a} = (a_1, \dots, a_d)$ be an \mathcal{E} -basis of $(\tilde{\mathcal{E}} \otimes V)^{G_F}$ (we know that $(\tilde{\mathcal{E}} \otimes V)^{G_F}$ is an \mathcal{E} -vector space of dimension d by Theorem 4.11). Then we have a period matrix $A \in M_{d \times d}(\tilde{\mathcal{E}})$ satisfying

$$\mathbf{a} = A\mathbf{e}. \tag{P}$$

We may assume that $A \in M_{d \times d}(\mathcal{O}_{\tilde{\mathcal{E}}})$ by multiplying our basis \mathbf{a} by a power of p . If \mathbf{a} can be chosen so that A lies in $M_{d \times d}(\mathcal{O}_{\tilde{\mathcal{E}}^r})$, then it will follow that $(\tilde{\mathcal{E}}^r \otimes V)^{G_F}$ is an \mathcal{E}^r -vector space of dimension d . In Proposition 6.2 we prove that if $\rho \in \mathbf{Rep}_{\mathbb{Q}_p}^r(G_F)$ then A may be chosen to be in $M_{d \times d}(\mathcal{O}_{\tilde{\mathcal{E}}^r})$. Consequently $D^r(V)$ is a (ϕ, ∇) -module over \mathcal{E}^r of dimension d . This shows that all representations in $\mathbf{Rep}_{\mathbb{Q}_p}^r(G_F)$ are $\tilde{\mathcal{E}}^r$ -admissible. Now let $f : V_1 \rightarrow V_2$ be a morphism in $\mathbf{Rep}_{\mathbb{Q}_p}^r(G_F)$. Then we have a corresponding morphism $f_{\mathcal{E}} : D(V_1) \rightarrow D(V_2)$ by Theorem 4.11. Using Proposition 6.5 we know that extension of scalars functor $\mathbf{M}\Phi_{\mathcal{E}^r, \sigma}^{et, \nabla} \rightarrow \mathbf{M}\Phi_{\mathcal{E}, \sigma}^{et, \nabla}$ is fully faithful. In particular, we know from Proposition 6.5 that $f_{\mathcal{E}}$ is actually the base change of a morphism $f_{\mathcal{E}^r} : D^r(V_1) \rightarrow D^r(V_2)$. This means that D^r defines a functor from $\mathbf{Rep}_{\mathbb{Q}_p}^r(G_F)$ to $\mathbf{M}\Phi_{\mathcal{E}^r, \sigma}^{et, \nabla}$ that is fully faithful.

To show that D^r is essentially surjective when restricted to $\mathbf{Rep}_{\mathbb{Q}_p}^r(G_F)$ we fix a (ϕ, ∇) -module M over \mathcal{E}^r of dimension d . Choose a basis $\mathbf{a} = (a_1, \dots, a_d)$ of M . Proposition 6.4 tells us that M trivializes over $\tilde{\mathcal{E}}^r$. Thus $L_M = (M \otimes_{\mathcal{E}^r} \tilde{\mathcal{E}}^r)^{\phi=1}$ is a \mathbb{Q}_p -vector space of dimension d with a continuous action ρ_M of G_F . Let $\mathbf{e}_M = (e_{M,1}, \dots, e_{M,d})$ be a basis of L_M . The two bases are related by a period matrix

$$\mathbf{a} = A_M \mathbf{e}_M,$$

whose entries are in $\tilde{\mathcal{E}}^r$. Then Proposition 6.3 tells us that $\rho_M \in \mathbf{Rep}_{\mathbb{Q}_p}^r(G_F)$. It follows that D^r is essentially surjective when restricted to $\mathbf{Rep}_{\mathbb{Q}_p}^r(G_F)$.

Lemma 6.1. *For $n \geq 1$ we have*

$$(\mathcal{O}_{\tilde{\mathcal{E}}} \otimes_{\mathbb{Z}_p} L)^{G_F} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \mathbb{Z} \cong (\mathcal{O}_{\tilde{\mathcal{E}}} \otimes_{\mathbb{Z}_p} L/p^n L)^{G_F}. \tag{Q}$$

That is, any period of $L/p^n L$ lifts to a period of L .

Proof. The left side of (Q) naturally injects into the right side. Tensoring with $\mathcal{O}_{\mathcal{E}_{F_n}}$ gives an injection

$$f : ((\mathcal{O}_{\tilde{\mathcal{E}}} \otimes_{\mathbb{Z}_p} L)^{G_F} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \mathbb{Z}) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}_{F_n}} \hookrightarrow (\mathcal{O}_{\tilde{\mathcal{E}}} \otimes_{\mathbb{Z}_p} L/p^n L)^{G_F} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}_{F_n}}.$$

We will prove that f is also surjective. Since $\mathcal{O}_{\mathcal{E}_{F_n}}$ is faithfully flat over $\mathcal{O}_{\mathcal{E}}$ the isomorphism f descends to give (Q). There is an inclusion of $(\mathcal{O}_{\tilde{\mathcal{E}}} \otimes_{\mathbb{Z}_p} L/p^n L)^{G_F}$ in $(\mathcal{O}_{\tilde{\mathcal{E}}} \otimes_{\mathbb{Z}_p} L/p^n L)^{G_{F_n}}$. Since $(\mathcal{O}_{\tilde{\mathcal{E}}} \otimes_{\mathbb{Z}_p} L/p^n L)^{G_{F_n}}$ is an $\mathcal{O}_{\mathcal{E}_{F_n}}$ -module, this gives an injective map

$$g : (\mathcal{O}_{\tilde{\mathcal{E}}} \otimes_{\mathbb{Z}_p} L/p^n L)^{G_F} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}_{F_n}} \hookrightarrow (\mathcal{O}_{\tilde{\mathcal{E}}} \otimes_{\mathbb{Z}_p} L/p^n L)^{G_{F_n}}.$$

In particular $g \circ f$ is injective. We will prove that $g \circ f$ is surjective, which will imply that f is surjective.

Let \overline{e}_i be the image of e_i in $L/p^n L$. Since G_{F_n} acts trivially on $L/p^n L$, we see that

$$(\mathcal{O}_{\tilde{\mathcal{E}}} \otimes_{\mathbb{Z}_p} L/p^n L)^{G_{F_n}}$$

is a free $\mathcal{O}_{\mathcal{E}_{F_n}}/p^n$ -module with basis $\overline{e}_1, \dots, \overline{e}_d$. Similarly, let \overline{a}_i be the image of a_i in $(\mathcal{O}_{\tilde{\mathcal{E}}} \otimes_{\mathbb{Z}_p} L)^{G_F} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \mathbb{Z}$. Consider the reduction modulo p^n of the period equation (P):

$$\overline{\mathbf{a}} = \overline{A} \overline{\mathbf{e}}.$$

Since $\overline{\mathbf{a}}$ and $\overline{\mathbf{e}}$ are fixed by G_{F_n} we see that the same is true for \overline{A} . In particular, we find $\overline{A} \in M_{d \times d}(\mathcal{O}_{\mathcal{E}_{F_n}}/p^n)$. It follows that each e_i is contained in the image of $g \circ f$, which means $g \circ f$ is surjective. \square

Proposition 6.2. *If $\rho \in \mathbf{Rep}_{\mathbb{Q}_p}^r(G_F)$ then the period matrix A may be taken to have entries in $\mathcal{O}_{\tilde{\mathcal{E}}^r}$.*

Proof. Let $M = (\tilde{\mathcal{E}} \otimes L)^{G_F}$ be the (ϕ, ∇) -module corresponding to ρ . The period matrix A is unique up to multiplication by an element of $B \in GL_d(\mathcal{O}_{\tilde{\mathcal{E}}})$, which corresponds to choosing a different basis of M . Thus we need to prove that there exists $B \in GL_d(\mathcal{O}_{\tilde{\mathcal{E}}})$ such that $BA \in GL_d(\mathcal{O}_{\tilde{\mathcal{E}}^r})$.

To find B we will find $B_n \in GL_d(\mathcal{O}_{\tilde{\mathcal{E}}})$ for each $n \geq 1$ satisfying

$$B_n \equiv B_{n-1} \pmod{p^{n-1}},$$

$$B_n A \pmod{p^n} \in GL_d(\mathcal{O}_{\mathcal{E}_{F_n}^{r,c}}/p^n \mathcal{O}_{\mathcal{E}_{F_n}^{r,c}}), \quad (*)$$

for a well chosen $c > 0$. Then $B = \lim_{n \rightarrow \infty} B_n$ will lie in $GL_d(\mathcal{O}_{\tilde{\mathcal{E}}})$ and

$$BA \in GL_d(\mathcal{O}_{\tilde{\mathcal{E}}^{r,c}}) \subset GL_d(\mathcal{O}_{\tilde{\mathcal{E}}^r}).$$

Let $c_0 > 0$ satisfy $G^{p^{n^r} c_0} \subset G(n+1)$ for all n and take c large enough to satisfy

$$c > \frac{p^r c_0 + |w_1(T)|}{p^r - 1}.$$

We proceed inductively. We may take B_1 to be a power of T times the identity matrix. This is because $B_1 A \pmod{p}$ may be regarded as a matrix with coefficients in F^{alg} , so that multiplying the entries by a large enough power of T will land in $\mathcal{O}_{\tilde{\mathcal{E}}^{r,c}} = \mathcal{O}_{\tilde{\mathcal{E}}^{r,c}}/p \mathcal{O}_{\tilde{\mathcal{E}}^{r,c}}$. Note that $e_i \pmod{pL}$ is fixed by G_{F_1} . Since $B_1 A e_i \pmod{pL}$ is also fixed by G_{F_1} it follows that $B_1 A \pmod{p}$ is contained in $\mathcal{O}_{\mathcal{E}_{F_1}^{r,c}}/p \mathcal{O}_{\mathcal{E}_{F_1}^{r,c}}$.

Now assume the existence of B_n . We have the following exact sequence:

$$0 \rightarrow \mathcal{O}_{F_{n+1}} \rightarrow \mathcal{E}_{F_{n+1}}^{r,c,n}/p^{n+1} \mathcal{E}_{F_{n+1}}^{r,c,n} \rightarrow \mathcal{O}_{\mathcal{E}_{F_{n+1}}^{r,c}}/p^n \mathcal{O}_{\mathcal{E}_{F_{n+1}}^{r,c}} \rightarrow 0.$$

The surjective map is reduction modulo p^n and the kernel is $p^n \mathcal{E}_{F_{n+1}}^{r,c,n}/p^{n+1} \mathcal{E}_{F_{n+1}}^{r,c,n} \cong \mathcal{O}_{F_{n+1}}$. Applying the exact functor $M \rightarrow M \otimes_{\mathbb{Z}_p} L$ gives an exact sequence of $\mathbb{Z}_p[G_F]$ -modules.

$$0 \rightarrow \mathcal{O}_{F_{n+1}} \otimes_{\mathbb{Z}_p} L/pL \rightarrow \mathcal{E}_{F_{n+1}}^{r,c,n}/p^{n+1} \mathcal{E}_{F_{n+1}}^{r,c,n} \otimes_{\mathbb{Z}_p} L \rightarrow \mathcal{O}_{\mathcal{E}_{F_{n+1}}^{r,c}}/p^n \mathcal{O}_{\mathcal{E}_{F_{n+1}}^{r,c}} \otimes_{\mathbb{Z}_p} L \rightarrow 0.$$

As $G(n+1)$ acts trivially on each term in this sequence we regard it as a sequence of $\mathbb{Z}_p[H(n+1)]$ -modules. Taking $H(n+1)$ invariants gives the exact sequence

$$(\mathcal{E}_{F_{n+1}}^{r,c,n}/p^{n+1} \mathcal{E}_{F_{n+1}}^{r,c,n} \otimes L)^{H(n+1)} \rightarrow (\mathcal{O}_{\mathcal{E}_{F_{n+1}}^{r,c}}/p^n \mathcal{O}_{\mathcal{E}_{F_{n+1}}^{r,c}} \otimes L)^{H(n+1)} \xrightarrow{\delta} H^1(H(n+1), \mathcal{O}_{F_{n+1}} \otimes L/pL).$$

Let $\epsilon_i = B_n A e_i \in \mathcal{O}_{\tilde{\mathcal{E}}} \otimes L$. By our inductive hypothesis (*) we see that reducing ϵ_i modulo p^n gives an element $\bar{\epsilon}_i \in \mathcal{O}_{\mathcal{E}_{F_{n+1}}^{r,c}} / p^n \mathcal{O}_{\mathcal{E}_{F_{n+1}}^{r,c}} \otimes_{\mathbb{Z}_p} L$, which is Galois invariant. By assumption we know $H(n+1)^{p^{nr}c_0} = 0$. Lemma 5.5 then shows that $H^1(H(n+1), \mathcal{O}_{F_{n+1}} \otimes_{\mathbb{Z}_p} L/pL)$ is annihilated by $T^{p^{nr}c_0}$. Thus there exists $\bar{\delta}_i \in (\mathcal{E}_{F_{n+1}}^{r,c,n} / p^{n+1} \mathcal{E}_{F_{n+1}}^{r,c,n} \otimes_{\mathbb{Z}_p} L)^{H(n+1)}$ that reduces to $T^{p^{nr}c_0} \bar{\epsilon}_i$ modulo p^n . Let δ_i be a Galois invariant lifting of $\bar{\delta}_i$ in $\tilde{\mathcal{E}} \otimes L$, which exists by Lemma 6.1. We have the following relation

$$\frac{1}{T^{p^{nr}c_0}} \delta_i \equiv \epsilon_i \pmod{p^n}.$$

In particular, if $B_{n+1} \in GL_d(\mathcal{O}_{\mathcal{E}})$ is the unique invertible matrix such that $\frac{1}{T^{p^{nr}c_0}} \boldsymbol{\delta} = B_{n+1} A \mathbf{e}$, where $\boldsymbol{\delta} = (\delta_1, \dots, \delta_d)^T$, then $B_{n+1} \equiv B_n \pmod{p^n}$.

It remains to show the entries of B_{n+1} have the desired growth properties. We have Teichmüller expansions:

$$\begin{aligned} B_{n+1} A &= \sum_{i=0}^n [X_i] p^i + p^{n+1} S_{n+1} \\ B_n A &= \sum_{i=0}^{n-1} [X_i] p^i + p^n S_n, \end{aligned}$$

where the X_i are $d \times d$ matrices in F^{alg} . Since $\delta_i = T^{p^{nr}c_0} B_{n+1} A e_i$ reduces to an element of $\mathcal{E}_{F_{n+1}}^{r,c,n} / p^{n+1} \otimes_{\mathbb{Z}_p} L / p^{n+1}$, we know that $T^{p^{nr}c_0} B_{n+1} A \in M_{d \times d}(\tilde{\mathbf{A}}^{r,c,n})$. It follows that

$$T^{p^{nr}c_0} \sum_{i=0}^n [X_i] p^i \in \tilde{\mathbf{A}}^{r,c,n}.$$

We also have

$$\sum_{i=0}^{n-1} [X_i] p^i \in \tilde{\mathbf{A}}^{r,c,n},$$

by our inductive assumption that $B_n A \pmod{p^n}$ is in $GL_n(\mathcal{O}_{\mathcal{E}^{r,c}/p^n})$. Since multiplication by T preserves $\tilde{\mathbf{A}}^{r,c,n}$ by Lemma 4.9 we know that

$$T^{p^{nr}c_0} \sum_{i=0}^{n-1} [X_i] p^i \in \tilde{\mathbf{A}}^{r,c,n},$$

which gives $T^{p^{nr}c_0} [X_n] p^n \in \tilde{\mathbf{A}}^{r,c,n}$. Write $T^{p^{nr}c_0} [X_n] p^n = \sum [Y_i] p^i$ so that $Y_i = 0$ for $i < n$. Our definition of $\tilde{\mathbf{A}}^{r,c,n}$ tells us that $v(Y_n) \geq c - p^{(n-1)r}c - |w_1(T)|$. Therefore

$$v(X_n) \geq c - p^{(n-1)r}c - |w_1(T)| - p^{nr}c_0.$$

The inequality $c > \frac{p^{rc_0} + |w_1(T)|}{p^r - 1}$ gives $c > \frac{p^{nr}c_0 + |w_1(T)|}{p^{nr} - p^{(n-1)r}}$. Rearranging this inequality we see that

$$v(X_n) \geq c - p^{(n-1)r}c - |w_1(T)| - p^{nr}c_0 > c - p^{nr}c,$$

which shows

$$B_{n+1} A \pmod{p^{n+1}} \in GL_d(\mathcal{O}_{\mathcal{E}_{F_{n+1}}^{r,c}} / p^{n+1} \mathcal{O}_{\mathcal{E}_{F_{n+1}}^{r,c}}).$$

□

Proposition 6.3. *If the period matrix A satisfying*

$$\mathbf{a} = A \mathbf{e},$$

lies in $M_{d \times d}(\mathcal{O}_{\tilde{\mathcal{E}}^r})$, then ρ lies in $\mathbf{Rep}_{\mathbb{Q}_p}^r(G_F)$.

Proof. The period matrix has a Teichmuller expansion:

$$A = \sum_{n=0}^{\infty} [A_n] p^n,$$

with $A_n \in M_{d \times d}(\mathcal{O}_{\widehat{F^{alg}}})$. By multiplying A by a power of $T^{-1}I_d$ we may assume that $v(A_0) = 0$. The condition $A \in M_{d \times d}(\mathcal{O}_{\mathcal{E}^r})$ guarantees the existence of $c > 0$ such that

$$v(A_n) \geq -p^{(n+1)r}c. \quad (3)$$

We will prove inductively two things: The first is that $\mu_n < p^{rn}c_0$ where

$$c_0 > \max \left(\frac{p^r p^{d^2} c}{p^r - 1}, \mu_1 \right). \quad (4)$$

The second is that $A_{n-1} \in \widehat{F_n^{perf}}$. The bound on μ_n will imply $\rho \in \mathbf{Rep}_{\mathbb{Q}_p}^r(G_F)$.

When $n = 1$ we have $\mu_1 < p^r c_0$ by our definition of c_0 . To see that $A_0 \in F_1^{perf}$ consider the action of G_{F_1} on $A\mathbf{e}$ reduced modulo p . Since G_{F_1} acts trivially on L/pL and it fixes $A\mathbf{e}$, we know that G_{F_1} must fix $A \bmod p$. This shows that A_0 is contained in $\widehat{F_1^{alg}}^{G_{F_1}} = \widehat{F_1^{perf}}$, which concludes our base case. Now assume the result for $n > 1$. Since $G_{F_{n+1}}$ fixes $A\mathbf{e}$ and acts trivially on $L/p^{n+1}L$ we see that $G_{F_{n+1}}$ fixes $A \bmod p^{n+1}$, which gives $A_n \in \widehat{F_{n+1}^{alg}}^{G_{F_{n+1}}} = \widehat{F_{n+1}^{perf}}$.

It remains to show that $\mu_{n+1} < p^{r(n+1)}c_0$. Let g be a nonzero element of $H(n+1)^{\mu_{n+1}}$. Then g has order p and if we view $H(n+1)$ as a subgroup of $GL_d(\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)$ through ρ we may represent g by a matrix $I_d + p^n B$ for some $B \in M_{d \times d}(\mathbb{Z}_p)$. The equation

$$g(A\mathbf{e}) - A\mathbf{e} \equiv 0 \bmod p^{n+1},$$

thus becomes

$$\left(\sum_{i=0}^n [g(A_i)] p^r \right) (I_d + p^n B) \mathbf{e} - \left(\sum_{i=0}^n [A_i] p^r \right) \mathbf{e} \equiv 0 \bmod p^{n+1}.$$

Since $g(A_i) = A_i$ for $i < n$ this yields the equation

$$[g(A_n)] p^n - [A_n] p^n \equiv B[A_0] p^n \bmod p^{n+1}.$$

In particular we have $v(g(A_n) - A_n) = 0$. Lemma 5.2 then gives

$$0 \geq v(A_n) + \frac{\lambda_{n+1} - 1}{|H(n+1)|}.$$

By applying Inequality (3) we see that $|H(n+1)| p^{(n+1)r} c > \lambda_{n+1}$. If we view G_{F_n} and $G_{F_{n+1}}$ as subgroups of $GL_d(\mathbb{Z}_p)$ it is clear that $\frac{|H(n+1)|}{|H(n)|} = |G_{F_n}/G_{F_{n+1}}| \leq p^{d^2}$. Then Lemma 5.3 and our inductive hypothesis that $\mu_n \leq p^{rn}c_0$ gives

$$\begin{aligned} \mu_{n+1} &\leq \frac{\lambda_{n+1}}{|H(n)|} + \mu_n \\ &\leq \frac{|H(n+1)|}{|H(n)|} p^{(n+1)r} c + \mu_n \\ &\leq p^{(n+1)r} c p^{d^2} + p^{rn} c_0. \end{aligned}$$

However the inequality (4) implies

$$p^{(n+1)r} c p^{d^2} + p^{rn} c_0 < p^{(n+1)r} c_0,$$

from which we see $\mu_{n+1} < p^{(n+1)r} c_0$. □

Proposition 6.4. *Let M be a (ϕ, ∇) -module over \mathcal{E}^r . Then M trivializes after base changing to $\tilde{\mathcal{E}}^r$.*

Proof. Let a_1, \dots, a_d be a basis of M and let C be the matrix of ϕ with respect to this basis. By Fontaine's theory of ϕ -modules we know that

$$M \otimes_{\mathcal{E}^r} \tilde{\mathcal{E}} \cong \bigoplus_{i=1}^d e_i \tilde{\mathcal{E}}.$$

The right hand side is the (ϕ, ∇) -module given by $\phi(e_i) = e_i$ and $\nabla(e_i) = 0$. Let $A \in GL_d(\tilde{\mathcal{E}})$ be the period matrix satisfying $\mathbf{a} = A\mathbf{e}$, where $\mathbf{a} = (a_1, \dots, a_d)$ and $\mathbf{e} = (e_1, \dots, e_d)$. It is enough to show that A has entries in $\tilde{\mathcal{E}}^r$. We have $\phi(\mathbf{a}) = C\mathbf{a}$ and $\phi(\mathbf{a}) = \phi(A\mathbf{e}) = \sigma(A)A^{-1}\mathbf{a}$. This gives $CA = \sigma(A)$. Since $w_n(\sigma(A)) = pw_n(A)$ we have

$$pw_n(A) = w_n(CA) \geq \min_{i+j \leq n} w_i(C) + w_j(A) \geq w_n(C) + w_n(A).$$

Therefore $w_n(A) \geq \frac{w_n(C)}{p-1}$. Since C is contained in $M_{d \times d}(\tilde{\mathbf{A}}^r)$, we see that the same is true of A . Therefore the entries of A are contained in $\tilde{\mathbf{A}}^r \cap \tilde{\mathcal{E}}$. □

Proposition 6.5. *Let M and N be (ϕ, ∇) -modules over \mathcal{E}^r and let $f_{\mathcal{E}} : M \otimes_{\mathcal{E}^r} \mathcal{E} \rightarrow N \otimes_{\mathcal{E}^r} \mathcal{E}$ be a morphism of (ϕ, ∇) -modules over \mathcal{E} . Then $f_{\mathcal{E}}$ descends to a morphism $f_{\mathcal{E}^r} : M \rightarrow N$ defined over \mathcal{E}^r .*

Proof. Let e_1, \dots, e_m be a basis of M and let f_1, \dots, f_n be a basis of N . Let A (resp. B) be the matrix for the Frobenius of M (resp. N) with respect to e_1, \dots, e_m (resp. f_1, \dots, f_n). Let $S \in M_{m \times n}(\mathcal{E})$ be the matrix of f with respect to these bases. We have to show that the entries of S lie in \mathcal{E}^r . The compatibility of f with Frobenius gives

$$B^{-1}SA = S^{\sigma},$$

where S^{σ} denotes σ applied to each entry of S . Let d be large enough so that

$$w_n(B^{-1}), w_n(A) \geq -p^{rn}d.$$

Then we have

$$\begin{aligned} w_n(S^{\sigma}) &\geq \min_{i+j+k=n} \{w_i(B^{-1}) + w_j(S) + w_k(A)\} \\ &\geq -2p^{rn}d + w_n(S). \end{aligned}$$

Since $w_n(S^{\sigma}) = pw_n(S)$ we see that $w_n(S) \geq \frac{-2p^{rn}d}{p-1}$, which means that S has entries in $\tilde{\mathbf{B}}^r$. Since the entries of S are also in \mathcal{E} we know from Lemma 4.7 that S has entries in \mathcal{E}^r . □

Remark 6.6. *The proof of Proposition 6.5 can be adapted to show that the base change functor from $M\Phi_{\mathcal{E}^{\dagger}, \sigma}^{et, \nabla}$ to $M\Phi_{\mathcal{E}, \sigma}^{et, \nabla}$ is fully faithful. This was first proved by Tsuzuki (see [23]).*

7 Global r -log-decay F -isocrystals and genus growth

To define overconvergent F -isocrystals, Berthelot defines a sheaf of rings $j^{\dagger}\mathcal{O}_{\mathcal{X}^{an}}$ on \mathcal{X}^{an} ([4, Chapter 2]). The sheaf $j^{\dagger}\mathcal{O}_{\mathcal{X}^{an}}$ agrees with the structure sheaf $\mathcal{O}_{\mathcal{Y}^{an}}$ when restricted to \mathcal{Y}^{an} . On any strict neighborhood $\mathcal{Y}^{an} \subset V$ the sections $\Gamma(V, j^{\dagger}\mathcal{O}_{\mathcal{X}^{an}})$ consist of functions on \mathcal{Y}^{an} that overconverge into each disc $|x| < |D|$. Additionally, for any open neighborhood V contained in $|D|$ there are no sections of $j^{\dagger}\mathcal{O}_{\mathcal{X}^{an}}$. An F -isocrystal on $\mathcal{O}_{\mathcal{Y}^{an}}$ is then overconvergent if it extends to an F -isocrystal on $j^{\dagger}\mathcal{O}_{\mathcal{X}^{an}}$. Following Crew ([6, Section 4]) one may *localize* at each $x \in D$ to obtain an F -isocrystal defined over \mathcal{E}^{\dagger} . In this section we will define a sheaf of rings

$\mathcal{O}_{\mathcal{Y}^{an}}^r$ on \mathcal{X}^{an} that will capture the property of r -log-decay at each $x \in D$. We will prove that an F -isocrystal over $\mathcal{O}_{\mathcal{Y}^{an}}$ extends to an F -isocrystal over $\mathcal{O}_{\mathcal{Y}^{an}}^r$ if and only if at each point $x \in D$ the localized F -isocrystal can be defined over \mathcal{E}^r . Finally, we will use the Riemann-Hurwitz-Hasse formula to relate the r -log-decay property to genus growth bounds for pro- p towers of curves over X . This is all summarized in Theorem 7.2.

7.1 The r -log-decay sheaves $\mathcal{O}_{\mathcal{Y}^{an}}^r$

Let $x \in D$ and let t_x be a rational function on X with a simple zero at x (i.e. t_x is a local parameter of x). Then t_x lifts to a function T_x , which is defined on an open subset of \mathcal{X} , and $\hat{\mathcal{O}}_{\mathcal{X},x}$ is isomorphic to $\mathcal{O}_K[[T_x]]$. Let $\underline{\mathcal{E}}_{T_x}|_{]x[}$ be the constant sheaf of the ring \mathcal{E}_{T_x} on the tube $]x[$ over x . Consider the immersions:

$$\begin{aligned} i_x &:]x[\hookrightarrow \mathcal{X}^{an} \\ j^{an} &: \mathcal{Y}^{an} \hookrightarrow \mathcal{X}^{an}. \end{aligned}$$

We will describe a map

$$f_x : j_*^{an} \mathcal{O}_{\mathcal{Y}^{an}} \rightarrow (i_x)_* (\underline{\mathcal{E}}_{T_x}|_{]x[}).$$

Let \mathcal{V}^{an} be a connected affinoid subspace of \mathcal{X}^{an} . If \mathcal{V}^{an} is contained entirely in $]x[$ (resp. \mathcal{Y}^{an}) then the target (resp. the source) is empty and there is nothing to describe. When \mathcal{V}^{an} has a nontrivial intersection with both \mathcal{V}^{an} and $]x[$, the map f_x will roughly send a section of $\Gamma(\mathcal{V}^{an}, j_*^{an} \mathcal{O}_{\mathcal{Y}^{an}}) = \Gamma(\mathcal{V}^{an} \cap \mathcal{Y}^{an}, \mathcal{O}_{\mathcal{Y}^{an}})$ to its T_x -adic expansions. Let us make this more precise. By Lemma 4.3 in [6], there exists r between 0 and 1 such that \mathcal{V}^{an} contains the annulus defined by $r \leq |T_x|_p < 1$. Let $\mathcal{R}_{[r,1)}$ denote the ring of analytic functions on $r \leq |T_x|_p < 1$. In particular we have a restriction map $\Gamma(\mathcal{V}^{an}, \mathcal{O}_{\mathcal{X}^{an}}) \rightarrow \mathcal{R}_{[r,1)}$. The ring $\mathcal{R}_{[r,1)}$ consists of Laurent series

$$\sum_{n=-\infty}^{\infty} a_n T_x^n$$

with coefficients in K such that $\sup |a_n|_p r_0^n < \infty$ for all $r \leq r_0 < 1$. The subring $\mathcal{R}_{[r,1)}^b \subset \mathcal{R}_{[r,1)}$ of bounded analytic functions consists of those Laurent series in $\mathcal{R}_{[r,1)}$ where the a_n are bounded. By the maximum modulus principle the functions in $\Gamma(\mathcal{V}^{an}, \mathcal{O}_{\mathcal{X}^{an}})$ are bounded, which means the image of the restriction map lies in $\mathcal{R}_{[r,1)}^b$. Putting this together gives

$$\begin{array}{ccc} \Gamma(\mathcal{V}^{an}, \mathcal{O}_{\mathcal{X}^{an}}) & \longrightarrow & \Gamma(\mathcal{V}^{an}, j_*^{an} \mathcal{O}_{\mathcal{Y}^{an}}) = \Gamma(\mathcal{V}^{an}, \mathcal{O}_{\mathcal{X}^{an}}) \langle T_x^{-1} \rangle \\ \downarrow & & \downarrow \\ \mathcal{R}_{[r,1)}^b & \longrightarrow & \mathcal{E}_{T_x}. \end{array}$$

This is essentially the localization process that Crew describes in [6, Section 4], which allows us to pass from global F -isocrystals on $\mathcal{O}_{\mathcal{Y}^{an}}$ to local F -isocrystals on \mathcal{E} . We would like to *pick out* the sections of $\Gamma(\mathcal{V}^{an}, j_*^{an} \mathcal{O}_{\mathcal{Y}^{an}})$ that are sent to $\mathcal{E}_{T_x}^r$ by f_x . To do this we utilize the constant sheaf $\underline{\mathcal{E}}_{T_x}^r|_{]x[}$ of the ring $\mathcal{E}_{T_x}^r$ on $]x[$. Consider the pullback $\mathcal{O}_{\mathcal{Y}^{an},x}^r$ that makes the following diagram Cartesian

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{Y}^{an},x}^r & \longrightarrow & j_*^{an} \mathcal{O}_{\mathcal{Y}^{an}} \\ \downarrow & & \downarrow f_x \\ (i_x)_* (\underline{\mathcal{E}}_{T_x}^r|_{]x[}) & \longrightarrow & (i_x)_* (\underline{\mathcal{E}}_{T_x}|_{]x[}) \end{array}$$

Taking the sections of this diagram on \mathcal{V}^{an} gives the Cartesian diagram

$$\begin{array}{ccc}
\Gamma(\mathcal{V}^{an}, \mathcal{O}_{\mathcal{Y}^{an}, x}^r) & \longrightarrow & \Gamma(\mathcal{V}^{an}, j_*^{an} \mathcal{O}_{\mathcal{Y}^{an}}) \\
\downarrow & & \downarrow \\
\mathcal{E}_{T_x}^r & \longrightarrow & \mathcal{E}_{T_x}.
\end{array}$$

In particular, we see that $\Gamma(\mathcal{V}^{an}, \mathcal{O}_{\mathcal{Y}^{an}, x}^r)$ consists of the functions on $\mathcal{V}^{an} \cap \mathcal{Y}^{an}$ that have r -log-decay around x . We remark that by Lemma 2.3 the construction of $\mathcal{O}_{\mathcal{Y}^{an}, x}^r$ does depend on the parameter T_x .

Finally, we define $\mathcal{O}_{\mathcal{Y}^{an}}^r$ by following this construction simultaneously for each point $x \in D$. Consider the sheaves on \mathcal{X}^{an}

$$\begin{aligned}
\underline{\mathcal{E}}_D &:= \bigoplus_{x \in D} (i_x)_* (\mathcal{E}_{T_x}^r|_{]x[}) \\
\underline{\mathcal{E}}_D^r &:= \bigoplus_{x \in D} (i_x)_* (\mathcal{E}_{T_x}^r|_{]x[})
\end{aligned}$$

and the map of sheaves

$$f_D := \bigoplus_{x \in D} f_x : j_*^{an} \mathcal{O}_{\mathcal{Y}^{an}} \rightarrow \underline{\mathcal{E}}_D.$$

Then $\mathcal{O}_{\mathcal{Y}^{an}}^r$ is taken to be the sheaf that completes the Cartesian diagram:

$$\begin{array}{ccc}
\mathcal{O}_{\mathcal{Y}^{an}}^r & \longrightarrow & j_*^{an} \mathcal{O}_{\mathcal{Y}^{an}} \\
\downarrow & & \downarrow f_D \\
\underline{\mathcal{E}}_D^r & \longrightarrow & \underline{\mathcal{E}}_D.
\end{array}$$

Remark 7.1. *If we were to replace $\mathcal{E}_{T_x}^r$ with $\mathcal{E}_{T_x}^\dagger$ we would recover Berthelot's overconvergent sheaf. However, Berthelot's construction avoids choosing any local parameters and extends to more general rigid varieties. It would be interesting to have a less ad-hoc construction of $\mathcal{O}_{\mathcal{Y}^{an}}^r$ that does not involve choosing any parameters and that works for higher dimensional varieties.*

7.2 Global results and genus growth

To state our main theorem we must explain how a p -adic representation of $\pi_1(Y)$ gives rise to a pro- p tower of curves. Let $\rho : \pi_1(Y) \rightarrow GL_d(\mathbb{Z}_p)$ be a continuous p -adic representation. We define $G = \text{Im}(\rho)$ and let

$$G = G(0) \supset G(1) \supset G(2) \supset \dots$$

be the p -adic Lie filtration given by $G(n) = \ker(G \rightarrow GL_d(\mathbb{Z}_p/p^n\mathbb{Z}_p))$. This gives rise to a pro- p tower of curves

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \dots$$

that is étale outside of D . We define g_n to be the genus of X_n and we define d_n to be the degree of X_n over X (i.e. $d_n = |G/G(n)|$). Our global interpretation of the r -log-decay property has to do with the asymptotic growth of $\frac{g_n}{d_n}$.

Theorem 7.2. *The following categories are equivalent*

1. *The category of continuous p -adic representations*

$$\rho : \pi_1(Y) \rightarrow GL_n(\mathbb{Q}_p)$$

such for each $x \in D$ with decomposition group G_{F_x} the restriction $\rho|_{G_{F_x}}$ lies in $\mathbf{Rep}_{\mathbb{Q}_p}^r(G_{F_x})$.

2. The category of continuous p -adic representations

$$\rho : \pi_1(Y) \rightarrow GL_n(\mathbb{Q}_p)$$

such that any pro- p tower of curves obtained from a $\pi_1(Y)$ -stable lattice as above has the property that $\frac{g_n}{d_n}$ grows $O(p^{n(r+1)})$.

3. The category of unit-root convergent F -isocrystals \mathcal{M} on \mathcal{Y}^{an} such that for each $x \in D$ the localized F -isocrystal M_x over \mathcal{E}_{T_x} descends to an F -isocrystal over $\mathcal{E}_{T_x}^r$. The morphisms are just morphisms of F -isocrystals.
4. The category of unit-root convergent F -isocrystals M on \mathcal{Y}^{an} that extend to $\mathcal{O}_{\mathcal{Y}^{an}}^r$.

Proof. The equivalence of (1) and (3) follows from applying Theorem 1.2 to each point $x \in D$. To prove the equivalence of (1) and (2) we will use the Riemann-Hurwitz-Hasse formula. Fix a \mathbb{Z}_p -lattice that is invariant under $\pi_1(Y)$ and consider the setup described above with the same notation. For $x \in D$ let $x_n \in X_n$ be a compatible system of points over x . We define F_{x_n} to be the fraction field of $\hat{\mathcal{O}}_{X_n, x_n}$, which gives a tower of local fields

$$F_{x_0} \subset F_{x_1} \subset F_{x_2} \subset \dots$$

whose union we denote F_{x_∞} . The Riemann-Hurwitz-Hasse formula (see [19, Theorem 6.1]) states

$$g_n - 2 = d_n(g_0 - 2) + \sum_{x' \in X_n} \delta_{x'},$$

where $\delta_{x'}$ denotes the different of the map $X_n \rightarrow X$ at x' . There are $\frac{d_n}{|F_{x_n} : F_x|}$ points of X_n above $x \in D$ and each point gives the same different. This gives

$$\frac{g_n}{d_n} = g_0 - 2 + \sum_{x \in D} \frac{\delta_{x_n}}{|F_{x_n} : F_x|} + \frac{2}{d_n}. \quad (\text{S})$$

If we have the local ramification bounds in the hypothesis of (1) we know from Lemma 5.6 that $\frac{\delta_{x_n}}{|F_{x_n} : F_x|}$ is $O(p^{rn})$ for each $x \in D$. It follows from (S) that $\frac{g_n}{d_n}$ is $O(p^{rn})$. This proves that (1) implies (2).

Conversely, assume that $\frac{g_n}{d_n}$ is $O(p^{rn})$. For $x \in D$ we know by (S) that $\frac{\delta_{x_n}}{|F_{x_n} : F_x|}$ grows $O(p^{rn})$. Let $c > 0$ such that $\frac{\delta_{x_n}}{|F_{x_n} : F_x|} \leq cp^{rn}$ for all n . Denote by μ_{x_n} the largest upper numbering ramification break of F_{x_n} over F_x . Equivalently we may define μ_{x_n} as the smallest number such that $G_{F_{x_\infty}/F_x}^{\mu_{x_n} + \epsilon} \subset G_{F_{x_\infty}/F_{x_n}}$ for any $\epsilon > 0$. We need to prove that μ_{x_n} is $O(p^{rn})$. By Corollary 5.8 we know that

$$\begin{aligned} cp^{rn} &\geq \frac{\delta_{x_n}}{|F_{x_n} : F_x|} \\ &\geq \frac{\mu_{x_n}}{|G_{F_{x_n}/F_x}^{\mu_{x_n}}|}. \end{aligned}$$

View $G_{F_{x_n}/F_x}$ as a subgroup of $GL_d(\mathbb{Z}_p/p^n\mathbb{Z}_p)$. The elements of $G_{F_{x_n}/F_x}^{\mu_{x_n}}$ all have order p , so that $g \in G_{F_{x_n}/F_x}^{\mu_{x_n}}$ corresponds to a matrix that reduces to the identity modulo p^{n-1} . This gives us the bound

$$|G_{F_{x_n}/F_x}^{\mu_{x_n}}| \leq p^{d^2},$$

from which we see

$$cp^{d^2} p^{rn} \geq \mu_{x_n}.$$

It remains to show the equivalence between (3) and (4). Let \mathcal{M}^r be an F -isocrystal of $\mathcal{O}_{\mathcal{Y}^{an}}^r$ -modules. Then we obtain an F -isocrystal \mathcal{M} of $\mathcal{O}_{\mathcal{Y}^{an}}$ -modules by restricting \mathcal{M}^r to \mathcal{Y}^{an} . By

our construction of $\mathcal{O}_{\mathcal{Y}^{an}}^r$ we know that \mathcal{M} will have the desired local properties. This gives a functor from (4) to (3). To see that this functor is fully faithful, we may follow Crew's argument in [6, 4.6-4.10] and then apply Proposition 6.5 (note that this is the same approach that Tsuzuki takes to proving the fully faithfulness of overconvergent F -isocrystals in [23, Theorem 5.1.1]). To prove that this functor is essentially surjective is the bulk of the work. Informally, what we need to prove that if a global F -isocrystal locally descends to an F -isocrystal with r -log-decay, then we can descend this F -isocrystal to have r -log-decay globally. Let \mathcal{M} be an F -isocrystal on \mathcal{Y}^{an} satisfying the conditions of (3). We will carefully choose a covering of \mathcal{X}^{an} by tubes and prove that \mathcal{M} globally has r -log-decay when restricted to each tube in our covering. These tubes will be chosen so that gluing the r -log-decay F -isocrystals is trivial.

For $x \in D$ we may find a rational function $\overline{f_x}$ on X such that $\overline{f_x}$ has a simple zero at x and $\overline{f_x}$ has a pole at each $x' \in D - \{x\}$. By allowing these poles to have high enough orders we may use Riemann-Roch to ensure that $\overline{f_x}$ and $\overline{f_{x'}}$ have no common zeros whenever $x \neq x'$. Let V_x be the largest Zariski open subset of X on which $\overline{f_x}$ is defined and so that the only zero of $\overline{f_x}$ on V_x is x (i.e. we take domain of definition of $\overline{f_x}$ and then remove all the zeros of $\overline{f_x}$ except for x). Since the $\overline{f_x}$ have no common zeros we have

$$X = \bigcup_{x \in D} V_x.$$

Also, by the way we chose the poles of $\overline{f_x}$ we have

$$V_x \cap V_{x'} \subset Y$$

whenever $x \neq x'$.

Let \mathcal{V}_x^{an} be $]V_x[$ and let \mathcal{W}_x^{an} be $\mathcal{V}_x^{an} -]x[$. Note that $\mathcal{W}_x^{an} \subset \mathcal{Y}^{an}$ because $V_x - \{x\}$ is contained in Y . We will let $\mathcal{O}_{\mathcal{V}_x^{an}}^r$ be the restriction of $\mathcal{O}_{\mathcal{X}^{an}}^r$ to \mathcal{V}_x^{an} . We will prove that the F -isocrystal $\mathcal{M}|_{\mathcal{W}_x^{an}}$ extends to an F -isocrystal \mathcal{M}_x^r of $\mathcal{O}_{\mathcal{V}_x^{an}}^r$ -modules. That is, when we restrict \mathcal{M}_x^r to \mathcal{W}_x^{an} we obtain an F -isocrystal of $\mathcal{O}_{\mathcal{W}_x^{an}}$ -modules isomorphic to $\mathcal{M}|_{\mathcal{W}_x^{an}}$ (recall that since $\mathcal{W}_x^{an} \subset \mathcal{Y}^{an}$ the restriction of $\mathcal{O}_{\mathcal{V}_x^{an}}^r$ to \mathcal{W}_x^{an} is $\mathcal{O}_{\mathcal{W}_x^{an}}$). We immediately see that \mathcal{M}_x^r and $\mathcal{M}_{x'}^r$ are isomorphic when restricted to $\mathcal{W}_x^{an} \cap \mathcal{W}_{x'}^{an}$, which means we can patch the \mathcal{M}_x^r together and obtain a sheaf \mathcal{M}^r of $\mathcal{O}_{\mathcal{Y}^{an}}^r$ -modules that restricts to \mathcal{M} on \mathcal{Y}^{an} .

Let \mathcal{V}_x be the formal open subscheme of \mathcal{X} corresponding to $V_x \subset X$ and let $\mathcal{W}_x \subset \mathcal{V}_x$ be the formal open subscheme corresponding to $V_x - \{x\}$. By [6, Lemma 2.5.1] we may find a small enough affine formal neighborhood $\mathcal{U}_x = \mathrm{Spf}(A_x)$ of x contained in V_x such that there is a free coherent sheaf on $\mathrm{Spf}(A_x) - \{x\} = \mathrm{Spf}(A_x \langle f_x^{-1} \rangle)$ whose rigid analytification is \mathcal{M} restricted to $\mathrm{Sp}(A_x \langle f_x^{-1} \rangle \otimes \mathbb{Q}) = \mathcal{U}_x^{an} -]x[$. In particular, if we let $\mathcal{T}_x = \mathrm{Spf}(A_x) - \{x\}$ then $\mathcal{M}|_{\mathcal{T}_x^{an}}$ is free. Note that \mathcal{U}_x^{an} is equal to the tube $]U_x[$ of an open set U_x containing x , which means that \mathcal{V}_x is covered by \mathcal{U}_x^{an} and \mathcal{W}_x^{an} . We will prove that $\mathcal{M}|_{\mathcal{T}_x^{an}}$ extends to a sheaf $\mathcal{M}_{\mathcal{U}_x}^r$ of $\mathcal{O}_{\mathcal{U}_x^{an}}^r$ -modules, utilizing the fact that $\mathcal{M}|_{\mathcal{T}_x^{an}}$ is free. Then $\mathcal{M}_{\mathcal{U}_x}^r$ and $\mathcal{M}|_{\mathcal{W}_x^{an}}$ are isomorphic on $\mathcal{T}_x \cap \mathcal{W}_x$ and we may glue them together to get a sheaf of $\mathcal{O}_{\mathcal{V}_x^{an}}^r$ -modules \mathcal{M}_x^r that extends $\mathcal{M}|_{\mathcal{W}_x^{an}}$.

We are now reduced to the case where $\mathcal{M}|_{\mathcal{T}_x^{an}}$ is free. In particular, if we let R_x be $\Gamma(\mathcal{T}_x^{an}, \mathcal{O}_{\mathcal{T}_x^{an}})$ then $\mathcal{M}|_{\mathcal{T}_x^{an}}$ corresponds to a free R_x -module M_x . Let T_x be a lifting of $\overline{f_x}$ in $\Gamma(\mathcal{W}_x, \mathcal{O}_{\mathcal{W}_x})$. The function T_x then defines a map \mathcal{W}_x^{an} to \mathbb{G}_m^{an} , where \mathbb{G}_m^{an} denotes the rigid analytic multiplicative group $\mathrm{Sp}(K \langle T_x, T_x^{-1} \rangle)$. Since \mathcal{T}_x^{an} is contained in \mathcal{W}_x^{an} we have a commutative diagram:

$$\begin{array}{ccc} K \langle T_x, T_x^{-1} \rangle & \longrightarrow & \mathcal{E}_{T_x} \\ \downarrow & & \downarrow \\ R_x & \longrightarrow & \mathcal{E}_{T_x}^r, \end{array}$$

where the top horizontal map is just the inclusion and the bottom horizontal map is the "expand in terms of T_x " map described in 7.1. We will show that there exists an F -isocrystal defined over $R_x \cap \mathcal{E}_{T_x}^r$ that becomes isomorphic to M_x after base-changing to R_x .

Our condition in (3) means that there is an F -isocrystal M_x^r defined over $\mathcal{E}_{T_x}^r$ such that

$$M_x \otimes_{R_x} \mathcal{E}_{T_x} \cong M_x^r \otimes_{\mathcal{E}_{T_x}^r} \mathcal{E}_{T_x}.$$

Let $\mathbf{e} = (e_1, \dots, e_n)$ be a basis of M_x and let $\mathbf{f} = (f_1, \dots, f_n)$ be a basis of M_x^r . Consider the matrix $L \in M_{n \times n}(\mathcal{E}_{T_x})$ satisfying $L\mathbf{e} = \mathbf{f}$. By Lemma 7.3 there exists $B \in M_{n \times n}(\mathcal{E}_{T_x}^r)$ such that BL has entries in $K\langle T_x, T_x^{-1} \rangle$. In particular, we can view BL as having entries in R_x . Since the Frobenius and connection matrices in terms of the basis \mathbf{e} are defined over R_x , the same is true for the basis $B\mathbf{e} = \mathbf{g}$. Similarly the Frobenius and connection matrices in terms of \mathbf{f} are defined over $\mathcal{E}_{T_x}^r$, the same is true for $B\mathbf{f} = \mathbf{g}$. It follows that the F -isocrystal structure in terms of the basis \mathbf{g} is defined over $\mathcal{E}_{T_x}^r \cap R_x$. The two bases \mathbf{e} and \mathbf{g} are related by a matrix in R_x , so we that the corresponding F -isocrystals are isomorphic over R_x . \square

Lemma 7.3. *Let $L \in GL_n(\mathcal{E})$. There exists $B \in GL_n(\mathcal{O}_K[[T]][T^{-1}, p^{-1}])$ such that BL has entries in $K\langle T, T^{-1} \rangle$.*

Proof. After multiplying L by powers of p and T we may assume that $L \in GL_n(\mathcal{O}_{\mathcal{E}})$ and that the reduction of L modulo p lies in $k[[T]]$. In particular, we may write

$$L(T) = \sum_{n=-\infty}^{\infty} l_n T^n,$$

where $l_n \in M_{d \times d}(\mathcal{O}_K)$. We let $v_p(l_n)$ denote the infimum of the p -adic valuation of the entries in l_n . Then L having entries in $\mathcal{O}_{\mathcal{E}}$ is equivalent to $v_p(l_n) \geq 0$ for all n and $v_p(l_n) \rightarrow \infty$ as $n \rightarrow -\infty$. Since

$$\mathcal{O}_K\langle T, T^{-1} \rangle \cong \varprojlim_{n \rightarrow \infty} \mathcal{O}_K/p^n \mathcal{O}_K[T, T^{-1}],$$

it will suffice to find $B \in GL_n(\mathcal{O}_K[[T]][T^{-1}, p^{-1}])$ such that BL reduces to a polynomial in T and T^{-1} modulo every power of p . We do this by successive p -adic approximation.

Let $k_0(T) \in GL_n(\mathcal{O}_K[[T]])$ be a matrix of power series that reduces to $L(T)^{-1}$ modulo p . This gives

$$k_0(T)L(T) \equiv 1 + T^{c_1} r_1(T)p \pmod{p^2},$$

where we may take $r_1(T)$ to be a matrix with entries in $\mathcal{O}_K[[T]]$ that is invertible in $\mathcal{O}_K[[T]]$ (i.e. T does not divide r_1 and the constant term is a p -adic unit). We may find a matrix $s_1(T)$ with entries in $\mathcal{O}_K[[T]]$ so that

$$a_1(T) = s_1(T) + T^{c_1} r_1(T)$$

has no terms of positive degree. Then $k_1(T) = 1 + s_1(T)p$ satisfies the congruence

$$k_1(T)k_0(T)L(T) \equiv 1 + a_1(T)p \pmod{p^2},$$

which lies in $\mathcal{O}_K/p^2 \mathcal{O}_K[T, T^{-1}]$. Following this pattern, we let $r_2(T)$ be a matrix with entries in $\mathcal{O}_K[[T]]$ such that

$$k_1(T)k_0(T)L(T) \equiv 1 + a_1(T)p + T^{c_2} r_2(T)p^2 \pmod{p^3}.$$

Then we find $s_2(T)$ with entries in $\mathcal{O}_K[[T]]$ so that

$$a_2(T) = s_2(T) + T^{c_2} r_2(T)$$

has no terms of positive degree. Setting $k_2(T) = 1 + s_2(T)p^2$ we see that $k_2(T)k_1(T)k_0(T)L(T)$ reduces modulo p^3 to an element of $\mathcal{O}_K/p^3 \mathcal{O}_K[T, T^{-1}]$. Continuing this process inductively we see that

$$B(T) = \prod_{i=0}^{\infty} k_i(T)$$

satisfies the desired properties. \square

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